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by

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ABSTRACT

Ehrenpreis' fundamental principle deals with spaces of entire functions and it can be applied to derive the Ehrenpreis-Martineau theorem which, by means of Fourier transformation, establishes the isomorphism between analytic functionals carried by bounded convex sets and entire functions of exponential type. In this paper the fundamental principle is derived for AU-spaces being the dual of spaces whose elements are the Fourier transforms of non-entire functions, and it is applied to derive extensions of the Ehrenpreis-Martineau theorem to analytic functionals carried by unbounded convex sets, the Fourier transforms of which are non-entire functions.

KEY WORDS & PHRASES: *Fourier transformation; analytic functionals carried by unbounded convex sets; holomorphic functions of several complex variables; Ehrenpreis-Martineau theorem; distributions, ultradistributions and Fourier hyper-functions; cohomology with bounds.*

1. INTRODUCTION

For some reasons a theorem is needed that characterizes the space of Fourier transforms of objects, such as distributions or analytic functionals, concentrated in a given unbounded convex set Ω in $\mathbb{C}^n$ [1]. When Ω is bounded and real, the Paley-Wiener-Schwartz theorem characterizes the space of Fourier transforms of distributions with compact support as a space of entire functions. When Ω is unbounded and real, this theorem can be extended (see VLADIMIROV [14]) so that it gives the space of Fourier transforms of tempered distributions with support in Ω as a space of functions holomorphic in the tubular radial domain $\mathbb{R}^n + i\,C$, where the open, convex cone C in $\mathbb{R}^n$ is determined by Ω. When Ω is bounded and complex, there is a theorem of Pólya for one variable and the extension to several variables of EHRENPREIS [2] and MARTINEAU [11] which characterizes the space of Fourier transforms of analytic functionals with carrier in Ω as a space of entire functions of exponential type. In this paper the last theorem is extended to the case of unbounded, convex, complex sets Ω. It turns out that, here too, the space of Fourier transforms of analytic functionals with unbounded carrier in Ω consists of functions of exponential type holomorphic in a tubular radial domain.

As in EHRENPREIS [3] the theorem for entire functions can be derived from Ehrenpreis' fundamental principle applied to the Cauchy-Riemann equations. These equations give rise to a multiplicity variety $\mathcal{W}$ of a very simple nature and, therefore, the local and semilocal theory is much easier than in the general case. For example, a semilocal extension of a function on $\mathcal{W}$ can be obtained by a similar process as in theorem 4.4.3 of HÖRMANDER [6]. In this paper the collection of semilocally defined functions is extended to a non-entire global function by theorem 5.1 and together

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[1] For example, writing formally the Newton interpolation series for functions in several variables and taking the Fourier transform of this series one comes to a domain of convergence which is an unbounded convex set in $\mathbb{C}^n$. Hence the series is valid for functions being the Fourier transform of objects concentrated in this set, see DE ROEVER [12].

with theorem 5.2 this yields the above mentioned generalization of the Ehrenpreis-Martineau theorem. However, for the proof of the general fundamental principle for non-entire functions no further work has to be done. For, the local and semilocal theory of EHRENPREIS [3] remains unchanged, while the extension to global results is obtained from theorems 5.1 and 5.2, which are the main theorems of this paper.

Examples of localizable spaces, i.e. spaces the fundamental principle is valid in, are obtained as spaces of objects, such as distributions, ultradistributions or Fourier hyperfunctions, which are the Fourier transforms of analytic functionals in Ω. These objects are given as boundary values of certain functions f holomorphic in tubular radial domains $\mathbb{R}^n + i\,C$. The analytic functionals are carried by Ω with respect to some class of weight functions determined by the behaviour of f near the vertex of C and with respect to some class of neighborhoods of Ω. In the case of distributions or Fourier hyperfunctions these neighborhoods can be obtained to be ε-neighborhoods and in the last case this extends a result in KAWAI [7], where a class of larger neighborhoods is considered.

As a particular case of the fundamental principle for non-entire functions one obtaines the fundamental principle for entire functions, now in the sense of theorem 3.1 of this paper, which is a weaker version than the one in EHRENPREIS [3]. On the other hand less restrictive conditions on the growth conditions are required, so that here the space $\mathcal{E}(O)$ with O convex is localizable (examples 8.1 and 8.2), where in EHRENPREIS [3] it is not; cf. remark 4.5 of [3]. The difference is that in [3] one globally defined function is obtained that satisfies all the bounds, while here for every bound a different global function is constructed with the aid of L^2-estimates for $\bar{\partial}$ of HÖRMANDER [6]. However, the representation of a solution in a localizable AU-space of a homogeneous system of differential equations with constant coefficients as Fourier transforms of measures concentrated on the zero set of the Fourier transform of the differential operator remains valid.

Besides in the application made here theorem 5.2, which is a generalization of theorem 7.6.11 of HÖRMANDER [6], is of interest by itself. Similarly to KOMATSU [9] this theorem together with the Ehrenpreis-Martineau theorems for non-entire functions (theorems 7.1, 7.2, 7.3 and 7.4) can be used to obtain resolutions of sheaves of solutions of systems of partial differential equations with constant coefficients in the spaces of objects being boundary values of functions of exponential type, holomorphic in tubular radial domains.

2. EHRENPREIS' FUNDAMENTAL PRINCIPLE

For a vector P of complex analytic polynomials EHRENPREIS [3] has defined a multiplicity variety $\mathcal{W}$ ($\mathcal{W}$ is defined in the set where all the components of P vanish). Let H be a space of entire functions satisfying certain growth conditions, then Ehrenpreis' fundamental principle says that H modulo $P\cdot\vec{H}$ is isomorphic to the space $H(\mathcal{W})$ of restrictions of entire functions to $\mathcal{W}$ satisfying the same growth conditions on $\mathcal{W}$ as restrictions to $\mathcal{W}$ of functions in H, provided that certain properties for the growth conditions hold. This is also true if P is a matrix of polynomials and if $\vec{\mathcal{W}}$ is an associated vector multiplicity variety, but then the restriction of an entire function to $\vec{\mathcal{W}}$ is defined only semilocally, i.e. in a priori given bounded sets covering $\mathbb{C}^n$. When the fundamental principle is valid, the space W of the Fourier transforms of the elements of the dual H' of H is called *localizable*; here, the Fourier transformation F is an isomorphism, thus $W = FH'$ or, equivalently, when W is the dual of some space W', $H = FW'$. [1)]

In some applications one rather starts with the space $H(\mathcal{W})$ than with H and the problem is to extend functions in $H(\mathcal{W})$ to functions in H. For example, in the Ehrenpreis-Martineau theorem (see §7) $\mathcal{W}$ is very simple,

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1) The following definition is used: when F is an isomorphism between the spaces B and $FB = A$, the Fourier transform of an element f in the dual A' of A is the element Ff of B' defined by

$$\langle Ff,\psi\rangle_B = \langle f,F\psi\rangle_A, \qquad \psi \in B.$$

By use of this definition the ambiguity mentioned on p.14 in EHRENPREIS [3] is avoided. Of course, as in [3] this definition corresponds to the following action of a function f, regarded as distribution in $\mathcal{D}'$, to testfunctions ϕ in $\mathcal{D}$

$$\langle f,\phi\rangle = \int f(x)\ \phi(x)dx.$$

but the extensions do not always agree with the formulation of the fundamental principle, as the following example shows:
let $H(\mathcal{W})$ be the space of entire functions f in $\mathbb{C}$ satisfying for every $\varepsilon > 0$

$$|f(z)| \le M_\varepsilon \exp \varepsilon |z|$$

and let H be the space of entire functions F in $\mathbb{C}^2$ satisfying for every $\varepsilon > 0$

$$|F(\theta)| \le M_\varepsilon (1+\|\theta\|)^m \exp \varepsilon \|\mathrm{Im}\ \theta\|,$$

where m depends on F; clearly the restriction to $\mathcal{W} = \{(\theta_1,\theta_2) \mid \theta_2 = i\theta_1\}$ of the growth conditions of H yield the growth conditions of $H(\mathcal{W})$; however, although $W = \mathcal{F}H'$ is localizable, it is not true that any function in $H(\mathcal{W})$ can be extended to a function in H; for example, the function

$$f(z) = \oint \exp(iz\zeta + 1/\zeta)\, d\zeta \in H(\mathcal{W})$$

cannot be written as $f(z) = F(z,iz)$ with $F \in H$, since all functions in H are polynomials (see VLADIMIROV [14]), while f is not.

Therefore, we analyse the formulation of the fundamental principle more precisely. An *analytically uniform structure* K for the space $W = \mathcal{F}H'$ is a collection of continuous positive functions k on $\mathbb{C}^n$, such that for each $F \in H$ and each $k \in K$

$$|F(z)| / k(z) \longrightarrow 0 \quad \text{as } \|z\| \to \infty$$

and such that the sets

$$\{F \in H \mid |F(z)| \le k(z),\ z \in \mathbb{C}^n\}$$

form a fundamental system of neighborhoods of zero in H. The space W is called an analytically uniform space (AU-space), see EHRENPREIS [3] p.9 (a)&(b) or BERENSTEIN & DOSTAL [1] p.7 (1)(iii). In order that W is localizable it is sufficient that besides the conditions [3] p.96 (a) & (b) or [1] p.8 (iv) H satisfies the following conditions ([3] p.96 (c) or

[1] p.8 (v)): there is a family M (BAU-structure) of continuous positive functions m on $\mathbb{C}^n$ with for every $m \in M$ and $k \in K$ $m(z) = O(k(z))$, such that the bounded sets

$$\{F \in H \mid |F(z)| \le \alpha\, m(z),\ z \in \mathbb{C}^n\}, \quad \alpha > 0,\ m \in M,$$

define a fundamental system of bounded sets in H; moreover, the functions $k \in K$ and $m \in M$ can be written as a product of functions k_i and m_i, respectively, in the variable z_i, $i = 1,\dots,n$ and these functions must satisfy certain conditions ([3] (4.3) & (4.4) or [1] p.21 (vii) & (viii)) among others ([1] (viii)): for every $\varepsilon > 0$ and every $m = m_1 \dots m_n \in M$, there exists $m^* = m_1^* \dots m_n^* \in M$, such that for each $i \in \{1,\dots,n\}$ and any $z_0 = x_0 + iy_0 \in \mathbb{C}$, there is an entire function $\phi(z)$ in $\mathbb{C}$ for which

$$\frac{m_i(z_0)\ |\phi(z)|}{\min\limits_{|\zeta - z_0| \le \varepsilon} |\phi(\zeta)|} \le m_i^*(z), \quad z \in \mathbb{C}. \tag{2.1}$$

In this case W is called *product localizable*, PLAU.

In the above example there are several possible AU-structures K on H, namely K_1 consisting of all functions $k(\theta) = k_1(\mathrm{Re}\,\theta_1)k_2(\mathrm{Im}\,\theta_1)k_1(\mathrm{Re}\,\theta_2)k_2(\mathrm{Im}\,\theta_2)$, where k_1 dominates all polynomials and where $k_2(y) = \exp \varepsilon|y|$, $\varepsilon > 0$, and K_2 consisting of all functions $k(\theta) = k_1(\mathrm{Re}\,\theta_1)k_1(\mathrm{Im}\,\theta_1)k_1(\mathrm{Re}\,\theta_2)k_1(\mathrm{Im}\,\theta_2)$. Possible BAU-structures M_1 belonging to K_1 and M_2 belonging to both K_1 or K_2 are: M_1 consisting of all functions $m(\theta) = m_1(\mathrm{Re}\,\theta_1)m_2(\mathrm{Im}\,\theta_1)m_1(\mathrm{Re}\,\theta_2)m_2(\mathrm{Im}\,\theta_2)$, where $m_1(x) = \alpha(1+|x|)^{\ell}$, $\alpha > 0$, $\ell > 0$ and where $m_2(y)$ is a continuous positive function which is dominated by every function $\exp \varepsilon|y|$, $\varepsilon > 0$, and M_2 consisting of all functions $m(\theta) = m_1(|\theta_1|)m_1(|\theta_2|)$.

M_1 does not satisfy condition (2.1), because m_2 is allowed to be a function that dominates all polynomials. Hence, if the AU-structure is defined by K_1 the BAU-structure has to be M_2 in order to let W be PLAU. However, if the topology of $H(\mathcal{W})$ is defined by restrictions to $\mathcal{W}$ of functions in K_1, M_2 does not induce a BAU-structure on $H(\mathcal{W})$. A BAU-structure on $H(\mathcal{W})$ would be the one induced by M_1.

The fact that M is a BAU-structure on $H(\mathcal{W})$ and condition (2.1) are needed to extend a collection of semilocally defined functions satisfying the bounds on W to a globally defined function in $\mathbb{C}^n$ satisfying the right bounds. Hence, if $H(W)$ is provided with the topology induced by one of the AU-structures K on H, one moreover has to require that the BAU-structure M on H belonging to K and satisfying the above conditions (among others condition (2.1)) is also a BAU-structure on $H(W)$. This assumption is omitted in BERENSTEIN & DOSTAL [1]. In EHRENPREIS [3] the topology of $H(W)$ is defined as the one induced by *all* the possible AU-structures on H. The special condition is satisfied then, but one has to know all the AU-structures on H.

In this paper we present the fundamental principle in a different way using L^2-estimates for $\bar{\partial}$ of HÖRMANDER [6]. Then the above mentioned problems are avoided and less restrictive conditions are required on the growth conditions for the functions in H. Another advantage of this approach is that it enables us to derive the principle for the space $E(O)$, where O is convex in $\mathbb{R}^n$, while the above method only yields that $E(O)$ is PLAU when O is a cube or that $E(O)$ is LAU when O is a convex polyhedron (see [3]). As far as the Ehrenpreis-Martineau theorem is concerned the fact that O should be a polyhedron is not serious, since an open convex set in $\mathbb{R}^n$ can be approximated arbitrarily close from the inside by polyhedra P and the theorem follows by application of the fundamental principle to the space $E(P)$. However, we will derive a similar theorem for analytic functionals carried by unbounded convex sets with respect to ε-neighborhoods and such sets cannot be approximated by polyhedra. The Fourier transforms of these analytic functionals are no longer entire and we need the fundamental principle for spaces H consisting of functions holomorphic in some open pseudoconvex set Ω and satisfying certain growth conditions there.

3. THE FUNDAMENTAL PRINCIPLE FOR NON-ENTIRE FUNCTIONS

We assume that the growth conditions on the functions of H can be expressed by L^p-norms with respect to weight functions of the form $\exp - \phi^\alpha$, where $\{\phi^\alpha\}_{\alpha\in A}$ are plurisubharmonic functions in Ω. It is stated that the generalization to arbitrary pseudoconvex open sets Ω does not consist of merely replacing $\mathbb{C}^n$ by Ω. Actually there are several possible ways for the behaviour of a function f on Ω:

Let $\{\Omega_k\}_{k=1}^{\infty}$ be an increasing sequence of relatively closed subsets of Ω with union Ω and let $\{\phi^\alpha\}_{\alpha\in A}$ be a decreasing net of plurisubharmonic functions in Ω. Denote for $p = 1,2,...$

(3.1) $$\|f\|_{\alpha,k}^{(p)} = [\int_{\Omega_k} |f(z)|^p \exp -p\phi^\alpha(z)\, d\lambda(z)]^{1/p},$$

with $\lambda(z)$ the Lebesgue measure in $\mathbb{C}^n$, and for $p = \infty$

$$\|f\|_{\alpha,k}^{(\infty)} = \sup_{z\in\Omega_k} |f(z)| \exp -\phi^\alpha(z);$$

we will write $\|\cdot\|_{\alpha,k}$ instead of $\|\cdot\|_{\alpha,k}^{(2)}$ when $p = 2$. Then there are the following possibilities for the behaviour of f:

(i) for some α and every $k = 1,2,...$ $\|f\|_{\alpha,k} \le K_k$

(ii) for each α fixed and every $k = 1,2,...$ $\|f\|_{\alpha,k} \le K_1(\alpha,k)$

(iii) for each k fixed and every $\alpha \in A$ $\|f\|_{\alpha,k} \le K_2(\alpha,k)$

where the constants K satisfy for every k and α

$$K_k \le K_{k+1},\quad K_1(\alpha,k) \le K_1(\alpha,k+1),\quad K_j(\alpha,k) \le K_j(\alpha',k),\ \alpha' \ge \alpha,\quad j = 1,2$$

and for every k and α there is an $\alpha_k \ge \alpha$ with

$$K_2(\alpha,k) \le K_2(\alpha_k,k+1).$$

When A is a countable set (ii) and (iii) are equal. If f is bounded with respect to the norm

$$\|f\|_{\alpha}^{(p)} = [\int_{\Omega} |f(z)|^p \exp -p\phi^\alpha(z)\, d\lambda(z)]^{1/p}$$

for $p = 1,2,...$ or

$$\|f\|_{\alpha}^{(\infty)} = \sup_{z\in\Omega} |f(z)| \exp -\phi^\alpha(z)$$

for $p = \infty$, we express this by saying that the sequence $\{K_k\}_{k=1}^{\infty}$ in (i) or for each $\alpha \in A$ the sequence $\{K_1(\alpha,k)\}_{k=1}^{\infty}$ in (ii) is bounded, i.e. $K_k \le K$ for every k and a certain constant K or $K_1(\alpha,k) \le K(\alpha)$ for every k and certain $K(\alpha)$ depending on α.

Ehrenpreis' fundamental principle deals with spaces of type (ii) where $\Omega_k = \mathbb{C}^n$ and $K_1(\alpha,k) \le K(\alpha)$. However, when Ω is not $\mathbb{C}^n$ we cannot work with spaces of functions of type (iii) and beginning with a space of functions of type (ii) we will always end with a space of functions of type (i). Therefore, although we are concerned with spaces of type (ii), we will change the formulation of the fundamental principle so that our calculations are performed in spaces of functions of type (i).

When all sets Ω_k are different we require that

$$(3.2) \qquad \forall k, \exists \ell > k : \forall z \in \Omega_k : z' \in B(z;\tfrac{1}{2},1) \Rightarrow z' \in \Omega_\ell,$$

where for $0 \le \delta < 1$ and $K \ge 0$

$$B(z;\delta,K) \overset{\text{def}}{=\!=} \{z' \mid \|z'-z\| \le \min[K,\delta d(z,\Omega^c)]\};$$

here $d(z,\Omega^c)$ denotes the distance from z to the complement of Ω, i.e.

$$d(z,\Omega) \overset{\text{def}}{=\!=} \inf_{z'\in\Omega^c} \|z-z'\|.$$

Moreover, there is a continuous plurisubharmonic function σ in Ω with

$$(3.3) \qquad \Omega_k = \{z \mid z \in \Omega,\ \sigma(z) \le k\}.$$

When the sets Ω_k are compact (3.3) is not a special condition on Ω (HÖRMANDER [6] theorem 2.6.7.ii), but we have in mind unbounded sets Ω_k. For $p = 1,2,\dots,\infty$ let

$$H_p(\Omega_k;\phi^\alpha)$$

be the Banach space of functions holomorphic in int Ω_k and in case $p = \infty$ also continuous on Ω_k, such that the norm (3.1) is finite and let

$$H_p[\Omega;\phi^\alpha] = \operatorname*{proj}_{k\to\infty} H_p(\Omega_k;\phi^\alpha).$$

When $p = 2$ we just write $H[\Omega;\phi^{\alpha}]$.

Finally we have to make one more assumption on the space H. Since it is not true that the neighborhood

$$\{B(z';\varepsilon,L) \mid z' \in B(z;\delta,K)\}$$

of $z \in \Omega$ again is contained in a neighborhood $B(z;\eta,M)$ of z for some η and M, we consider neighborhoods of z of the form

$$D(z;\delta,K) \overset{\text{def}}{=\!=} \{z' \mid z' \in \Omega,\ d(z',\Omega^c) \geq \delta d(z,\Omega^c),\ \|z-z'\| \leq K\},$$

$0 \leq \delta < 1$, $K \geq 0$. Then

$$B(z;1-\delta,K) \subset D(z;\delta,K) \tag{3.4}$$

and

$$\{D(z';\varepsilon,L) \mid z' \in D(z;\delta,K)\} \subset D(z;\varepsilon\delta,K+L).$$

So, if for $K \geq 0$ we write

$$S(z;K) \overset{\text{def}}{=\!=} D(z;e^{-K},K),$$

then

$$\{S(z';K) \mid z' \in S(z;L)\} \subset S(z;K+L).$$

For a plurisubharmonic function ϕ in Ω and for $N,M,K \geq 0$ let

$$\tilde{\phi}_{N,M,K}(z) \overset{\text{def}}{=\!=} \max\{\phi(z') + N\log(1+\|z'\|^2) + \log(1+d(z',\Omega^c)^{-M}) \mid z' \in S(z;K)\}. \tag{3.5}$$

When $\Omega = \mathbb{C}^n$, $S(z;K) = \{z' \mid \|z-z'\| \leq K\}$ and $\tilde{\phi}_{N,M,K}$ (which does not depend on M) again is plurisubharmonic. Like the third term in the sum occurring in (3.5) we will meet plurisubharmonic functions ψ (bounded far from $\partial\Omega$ and tending to infinity near $\partial\Omega$) that are convex functions of $d(z,\Omega^c)$. Then if in (3.5) $S(z;K)$ is replaced by $\{z' \mid z' \in \Omega,\ d(z',\Omega^c) \geq e^{-K} d(z,\Omega^c)\}$ $\tilde{\psi}_{N,M,K}$ again is plurisubharmonic. Therefore, in (3.6) we will change the function (3.5) slightly in order to get a plurisubharmonic function.

Now we require that the net $\{\phi^\alpha\}_{\alpha\in A}$ is such that for every N,M,K ≥ 0 and every $\alpha \in A$ there is an $\alpha' \geq \alpha$, a constant $A = A(\alpha,N,M,K)$ and a plurisubharmonic function $\phi^\alpha_{N,M,K}$ with

$$(3.6) \qquad \tilde{\phi}^{\alpha'}_{N,M,K}(z) \leq \phi^\alpha_{N,M,K}(z) \leq \tilde{\phi}^\alpha_{N,M,K}(z)+A, \qquad z \in \Omega$$

and such that for every N ≥ 0

$$(3.7) \qquad H \overset{\text{def}}{=\!=} \operatorname*{proj}_{\alpha\in A} H_p[\Omega;\phi^\alpha] = \operatorname*{proj}_{\alpha\in A} H_p[\Omega;\phi^\alpha_N],$$

where $\phi^\alpha_N \overset{\text{not}}{=\!=} \phi^\alpha_{N,N,N}$. Remark that for the above mentioned ψ we may take $\alpha' = \alpha$ in (3.6). Furthermore, we may choose $\phi^\alpha_{N,M,K}$ in (3.6) such that for every N ≥ 0 and M ≥ 0

$$(3.8) \qquad (\phi^\alpha_N)_M \leq \phi^\alpha_{N+M} + A$$

for some constant A depending on α, N and M.

Conditions (3.2) and (3.7) imply that H is independent of $p \in \{1,2,\ldots\infty\}$ (see WLOKA [16]) and that moreover, when $p = \infty$, for $f \in H$, $\alpha \in A$ and every k

$$(3.9) \qquad |f(z)|\exp-\phi^\alpha(z) \to 0 \quad \text{as} \quad z \to \partial\Omega \quad \text{or} \quad \|z\| \to \infty \text{ in } \Omega_k.$$

When $\Omega = \mathbb{C}^n$ condition (3.7) is just condition (b) p.196 in EHRENPREIS [3] or (iv) p.8 in BERENSTEIN & DOSTAL [1].

Let P be a p×q-matrix with polynomial entries and let $\vec{W}$ be an associated vector multiplicity. The fundamental principle for non-entire functions proved in this paper says that the map ρ

$$(3.10) \qquad \operatorname*{proj}_{\alpha\in A}\left\{ H[\Omega;\phi^\alpha]^p \Big/ H[\Omega;\phi^\alpha]^p \cap P\cdot H[\Omega;\phi^\alpha]^q \right\} \xrightarrow{\ \rho\ } \operatorname*{proj}_{\alpha\in A} H[\Omega\cap\vec{W};\phi^\alpha]$$

is an isomorphism. Here ρ is defined only semilocally if p > 1.

Less conditions are required, if we deal with the spaces $H = \operatorname{proj}_{\alpha\in A} H_p(\Omega;\phi^\alpha)$ or $H = \operatorname{ind}_{\ell\to\infty} \operatorname{proj}_{\alpha\in A} H_p(\Omega_\ell;\phi^\alpha)$, where $\{\Omega_\ell\}_{\ell=1}^\infty$ is a decreasing sequence of open pseudoconvex sets with intersection Ω and where $\{\phi^\alpha\}_{\alpha\in A}$ are plurisubharmonic functions in Ω_1. In fact, in the first case we only require that

$$(3.11) \qquad H \overset{\text{def}}{=\!=} \operatorname{proj}_{\alpha\in A} H_p(\Omega;\phi^\alpha) = \operatorname{proj}_{\alpha\in A} H_p(\Omega;\phi_N^\alpha)$$

for every $N \geq 0$ and in the second case we require that

$$(3.12) \qquad H \overset{\text{def}}{=\!=} \operatorname{ind}_{\ell\to\infty} \operatorname{proj}_{\alpha\in A} H_p(\Omega_\ell;\phi^\alpha) = \operatorname{ind}_{\ell\to\infty} \operatorname{proj}_{\alpha\in A} H_p(\Omega_\ell;\phi_N^\alpha),$$

where ϕ_N^α is defined by (3.5) and (3.6) with Ω replaced by Ω_1, and that

$$(3.13) \quad \forall\ell, \exists k>\ell, \exists\delta>0 \ : \ \forall z \in \Omega_k \ : \ \|z'-z\| \leq \min[1,\delta d(z,\Omega_1^c)] \Rightarrow z' \in \Omega_\ell.$$

Also here the spaces (3.11) and (3.12) are independent of $p \in \{1,2,\ldots,\infty\}$. Now the fundamental principle says that ρ is an isomorphism between

$$(3.14) \qquad \operatorname{proj}_{\alpha\in A}\left\{ H(\Omega;\phi^\alpha)^p \Big/ H(\Omega;\phi^\alpha)^p \cap P\cdot H(\Omega;\phi^\alpha)^q \right\} \xrightarrow{\ \rho\ } \operatorname{proj}_{\alpha\in A} H(\Omega\cap\vec{W};\phi^\alpha)$$

and

$$(3.15) \ \operatorname{ind}_{\ell\to\infty} \operatorname{proj}_{\alpha\in A}\left\{ H(\Omega_\ell;\phi^\alpha)^p \Big/ H(\Omega_\ell;\phi^\alpha)^p \cap P\cdot H(\Omega_\ell;\phi^\alpha)^q \right\} \xrightarrow{\ \rho\ } \operatorname{ind}_{\ell\to\infty} \operatorname{proj}_{\alpha\in A} H(\Omega_\ell\cap\vec{W};\phi^\alpha).$$

That ρ in (3.15) is an isomorphism follows from (3.12) and (3.13) and the fact that ρ in (3.14) is an isomorphism. So we will prove that (3.10) and (3.14) are isomorphisms and we obtain the following theorem.

THEOREM 3.1 *(Fundamental principle). Let* W *be a locally convex space which is the strong dual of some locally convex space* W' *such that* $H = FW'$ *is a locally convex space of holomorphic functions in some open pseudoconvex set* Ω *whose growth is determined by* $\exp \phi^\alpha$*, where* $\{\phi^\alpha\}_{\alpha\in A}$ *are plurisub-*

harmonic functions in Ω. *Here the Fourier transformation* $\mathcal{F}$ *is an isomorphism. If* H *satisfies* (3.11), *then* W *is localizable, i.e. the map* ρ *in* (3.14) *is an isomorphism or, if* Ω *satisfies* (3.2) *and* (3.3) *and if* H *satisfies* (3.7), *then the map* ρ *in* (3.10) *is an isomorphism. When* H *consists of holomorphic functions in open pseudoconvex neighborhoods* Ω_ℓ *of the open or closed set* Ω *satisfying* (3.13), *then the map* ρ *in* (3.15) *is an isomorphism, if* H *satisfies* (3.12).

We remark that (3.10), (3.14) and (3.15) is a weaker version of the fundamental principle than in EHRENPREIS [3], where ρ acts on a space of the form

$$\operatorname*{proj}_{\alpha\in A} H(\mathbb{C}^n;\phi^\alpha)^p \Big/ P\cdot \operatorname*{proj}_{\alpha\in A} H(\mathbb{C}^n;\phi^\alpha)^q .$$

However, the representation (theorem 7.1 [3] or (9) p.93 [1]) of a solution T in W of a system of homogeneous differential equations $\vec{P}(D)T = 0$, $\vec{P} = (P_1,\ldots,P_q)$, P_i polynomials, remains valid; namely, T can be written symbolically as

$$(3.16) \qquad T(x) = \sum_{j=1}^{r} \int_{V_j\cap\Omega_k} [\partial_j \exp iz\cdot x]\exp - \phi^\alpha(z)\, d\mu_j(z),$$

when H is given by (3.7), for some ϕ^α, some k and some totally bounded measures μ_j on the algebraic variety $\Omega_k \cap V_j \subset \Omega_k \cap \{z \mid P_1(z)=\ldots=P_q(z)=0\}$, or when H is given by (3.11) we set $\Omega_k = \Omega$ in (3.16). Similarly, when H is given by (3.12), (3.16) becomes

$$T(x) = \sum_{j=1}^{r} \int_{V_j\cap\Omega_\ell} [\partial_j \exp iz\cdot x]\exp - \phi^\alpha(z)\, d\mu_j^\ell(z)$$

for every $\ell = 1,2,\ldots$. Here V_j and the differential operators ∂_j in z are determined by $\vec{P}$. Indeed (cf. BERENSTEIN & DOSTAL [1]), let $T \in W$ be a solution of $\vec{P}(D)T = 0$ and let $f \in H$ be such that for all $\alpha \in A$

$$(3.17) \qquad f(z) = \vec{P}(z)\cdot \vec{g}^{\alpha}(z)$$

for some $\vec{g}^{\alpha} \in H[\Omega;\phi^{\alpha}]^{q}$, $\vec{g}^{\alpha} \in H(\Omega;\phi^{\alpha})^{q}$ or $\vec{g}^{\alpha} \in H(\Omega_{\ell_0};\phi^{\alpha})^{q}$, for some ℓ_0, respectively. Then $F^{-1}T$ belongs to some space $H[\Omega;\phi^{\alpha_0}_{N,0,0}]'$, $H(\Omega;\phi^{\alpha_0}_{N,0,0})'$ or every $H(\Omega_{\ell};\phi^{\alpha_\ell}_{N,0,0})$, $\ell = 1,2,\ldots$, where N is sufficiently large. If in the last case α_0 denotes α_{ℓ_0}, then

$$<F^{-1}T,f(z)> = <F^{-1}T,\vec{P}(z)\cdot\vec{g}^{\alpha_0}(z)> = <T,\vec{P}(-D)\cdot F^{-1}\vec{g}^{\alpha_0}> =$$

$$= <\vec{P}(D)T,F^{-1}\vec{g}^{\alpha_0}> = 0.$$

Conversely, any $T \in W$ with $<F^{-1}T,f(z)> = 0$ for f satisfying (3.17) satisfies

$$<\vec{P}(D)T,F^{-1}\vec{g}> = 0$$

for all functions $\vec{g}$ belonging to a certain $H[\Omega;\phi^{\alpha_0}]^{q}$, $H(\Omega;\phi^{\alpha_0})^{q}$ or $H(\Omega_{\ell_0};\phi^{\alpha_0})^{q}$, hence $\vec{P}(D)T = 0$ in W. The representation (3.16) follows from the isomorphism (3.10) and property (3.9), while the isomorphisms (3.14) and (3.15) yield the other cases. A similar representation holds for a vector solution $\vec{T}$ when P is a p×q-matrix (see EHRENPREIS [3] theorem 7.3).

The space $H[\Omega;\phi^{\alpha}]'$ arising here can be represented as follows: as a projective limit of Hilbert spaces $H[\Omega;\phi^{\alpha}]$ is an FS*-space (see KOMATSU [8]), therefore, it is reflexive and its strong dual equals

$$\operatorname{ind}_{k\to\infty} \bar{H}(\Omega_k;\phi^{\alpha})',$$

where $\bar{H}(\Omega_k;\phi^{\alpha})$ is the closure of $H[\Omega;\phi^{\alpha}]$ in the norm (3.1) for $p = 2$; hence bounded sets in $H[\Omega;\phi^{\alpha}]'$ are contained in some $\bar{H}(\Omega_k;\phi^{\alpha})'$.

In the above we assumed that the Fourier transformation F is an isomorphism between H' and W. So, at least the set of functions in z $\{e^{iz\cdot\zeta}\}_{\zeta\in O}$, where ζ ranges in some set $O \subset \mathbb{C}^n$, must be contained in H and, moreover, since F is injective from H' onto W this set should be dense in H. The space W then consists of objects (such as functions or distributions) in O. The set O is convex, since $\exp iz\cdot\zeta_1 \in H_\infty[\Omega;\phi^{\alpha}]$ or $H_\infty(\Omega;\phi^{\alpha})$ and $\exp iz\cdot\zeta_2 \in H_\infty[\Omega;\phi^{\alpha}]$ or $H_\infty(\Omega;\phi^{\alpha})$ imply $\exp iz\cdot(t\zeta_1+(1-t)\zeta_2) \in H_\infty[\Omega;\phi^{\alpha}]$ or $H_\infty(\Omega;\phi^{\alpha})$, respectively, $0 \le t \le 1$. On the other hand, if one starts with an AU-space W, which is the dual of W' of objects on O, then the set $\{e^{iz\cdot\zeta}\}_{z\in\Omega}$ of functions in ζ should be dense in W, since also F is an isomorphism between W' and H. From this it follows that in such cases Ω is convex too. However, nor this condition, nor the the fact that $\{e^{iz\cdot\zeta}\}_{\zeta\in O}$ is dense in H are used to derive the fundamental principle, as they are only needed when Fourier transformation comes in.

4. SEMILOCAL THEORY

Let $U = \{U_i\}_{i=1}^{\infty}$ be a special open covering of Ω and let $U^{(1)}$ be a special open shrinking of U. Then the proof in EHRENPREIS [3] shows that any function f in $\operatorname{proj}_{\alpha\in A} H[\Omega\cap\vec{W},\phi^{\alpha}]$ or $\operatorname{proj}_{\alpha\in A} H(\Omega\cap\vec{W},\phi^{\alpha})$ can be extended to a collection of functions c_i holomorphic in U_i and satisfying good bounds.

Let $C^p[U,F,\phi^{\alpha}]$ be the set of all alternating p-cochains c of the covering U with values in the sheaf F that satisfy for every k

$$(4.1)\qquad \|c\|_{\alpha,k} = \left[\sum_{|s|=p+1} \int_{U_s\cap\Omega_k} \|c_s(z)\|^2 \exp - 2\phi^{\alpha}(z)\, d\lambda(z)\right]^{\frac{1}{2}} < \infty,$$

where $\|f(z)\|^2 = |f_1(z)|^2+\dots+|f_q(z)|^2$ if $f = (f_1,\dots,f_q) \in A(U_s\cap\Omega_k)^q$; here $A(\Omega)$ denotes the set of holomorphic functions in Ω. The coverings U and $U^{(1)}$ have to satisfy certain properties listed in §6 in order that the estimates for cochains can be carried over to globally defined functions and conversely.

Let A be the sheaf in Ω of germs of holomorphic functions and let F be the image under P of the sheaf A^q: $F = P\cdot A^q$, a subsheaf of A^p. Let $C^s[U,A^p,\phi;P]$ be the set of cochains $c \in C^s[U,A^p,\phi]$ with

$$\delta c \in C^{s+1}(U,F).$$

In EHRENPREIS [3] it is shown that the restriction map $\tilde{\rho}^L$ is an isomorphism onto:

$$\operatorname{proj}_{\alpha\in A} C^0[U,A^p,\phi^{\alpha};P] \Big/ \operatorname{proj}_{\alpha\in A} C^0[U,A^p,\phi^{\alpha};P] \cap P\cdot\operatorname{proj}_{\alpha\in A} C^0[U^{(1)},A^q,\phi^{\alpha}] \xrightarrow{\tilde{\rho}^L}$$

$$\xrightarrow{\tilde{\rho}^L} \operatorname{proj}_{\alpha\in A} H[\Omega\cap\vec{W};\phi^{\alpha}].$$

Indeed, a function f in $\operatorname{proj}_{\alpha\in A} H[\Omega\cap\vec{W};\phi^{\alpha}]$ is extended to a collection of functions $\{c_i\}_{i=1}^{\infty}$ with c_i holomorphic in U_i. Firstly, for each i f is extended to a finite collection of functions holomorphic in finitely many very small sets covering U_i, whose difference on the overlaps are sections in F. Then one has to apply a piecing together process of this collection of functions to one function c_i in U_i. As is remarked in EHRENPREIS [3], this process follows the same lines as the proof of the similar statements for the map λ we will define and even it is more simple, because U_i is a bounded set, so that no convergence factors such as ϕ arising in condition (2.1) are needed. We have not assumed this condition, so that the proof of EHRENPREIS [3] is valid here too. Of course, one can also follow the piecing together process we perform in section 6.

In fact, each set U_i is covered in such a way that the bounds for c_i depend on the bounds for f on $V_i \cap \vec{W}$, where V_i is the enlargement by a factor 2 of U_i the center z_i kept fixed. Then the minimal size of the sets that cover U_i is proportional to a power of $(1+\|z_i\|)^{-1}$ and to a power of the size β_i of U_i. Also the maximal number of sets covering U_i is proportional to a power of $1+\|z_i\|$ and of β_i^{-1}. However, these powers do not depend on i, see EHRENPREIS [3] chapter III. It follows from our piecing together process that c_i satisfies for some N and K independent of i

$$\left[\int_{U_i} \|c_i(z)\|^2 \, d\lambda(z)\right]^{\frac{1}{2}} \le K\left(\frac{1+\|z_i\|^2}{\beta_i}\right)^N \sup_{z\in V_i\cap\vec{W}} \|f(z)\|.$$

Furthermore, the sets U_i will be such that they have a fixed size if they are far enough from $\partial\Omega$ or that the size is proportional to d_i, where d_i is the distance from U_i to $\partial\Omega$. Therefore, for every $\alpha \in A$ and some N

$$\left[\int_{U_i} \|c_i(z)\|^2 \exp - 2\phi_N^{\alpha}(z) \, d\lambda(z)\right]^{\frac{1}{2}} \le K \sup_{z\in V_i\cap\vec{W}} \|f(z)\| \exp - \phi^{\alpha'}(z),$$

where α' is determined by (3.6). Since the sets U_i can be chosen so that every $z \in \Omega$ is contained in not more than L different sets V_i and since V_i is contained in Ω_ℓ if $U_i \cap \Omega_k \neq \emptyset$ for some $\ell > k$, in virtue of (3.7) for every k and $\alpha \in A$ we get

$$(4.2)\qquad \|c\|_{\alpha,k} \leq LK \sup_{z\in\Omega_\ell\cap\vec{\mathcal{W}}} \|f(z)\| \exp - \phi^{\alpha'}(z).$$

A similar procedure, now with respect to the covering $\mathcal{U}^{(1)}$, shows that $\tilde{\rho}^L$ is injective, see EHRENPREIS [3]. Finally (4.2) implies that $(\tilde{\rho}^L)^{-1}$ is continuous.

However, if we use the following isomorphism, the proofs in the sequel will be considerably facilitated, while the final statements do not change:

$$\operatorname*{proj}_{\alpha\in A} C^0[\mathcal{U},A^p,\phi^\alpha;P] \Big/ \operatorname*{proj}_{\alpha\in A} C^0[\mathcal{U},A^p,\phi^\alpha;P] \cap P\cdot\operatorname*{proj}_{\alpha\in A} C^0[\mathcal{U}^{(1)},A^q,\phi^\alpha] \longrightarrow$$

$$\longrightarrow \operatorname*{proj}_{\alpha\in A}\left\{C^0[\mathcal{U}^{(1)},A^p,\phi^\alpha;P] \Big/ C^0[\mathcal{U}^{(1)},A^p,\phi^\alpha;P] \cap P\cdot C^0(\mathcal{U}^{(1)},A^q)\right\}.$$

This map is surjective and has continuous inverse (here each $U_i \in \mathcal{U}$ is covered by finitely many sets from $\mathcal{U}^{(1)}$, the number and size depending only on the size of U_i) and that it is injective can be seen as follows: any $c \in \operatorname*{proj}_{\alpha\in A} C^0[\mathcal{U},A^p,\phi^\alpha;P]$ that can be written as $c = P\cdot g$ with $g \in C^0(\mathcal{U}^{(1)},A^q)$ vanishes on $\Omega \cap \vec{\mathcal{W}}$, because also $\mathcal{U}^{(1)}$ is a covering of Ω, so that, since $\tilde{\rho}^L$ is injective, it can be written as $c = P\cdot g$ with $g \in \operatorname*{proj}_{\alpha\in A} C^0[\mathcal{U}^{(1)},A^q,\phi^\alpha]$. Similarly, it follows that $C^0[\mathcal{U}^{(1)},A^p,\phi^\alpha;P] \cap P\cdot C^0(\mathcal{U}^{(1)},A^q)$ is a closed subspace of $C^0[\mathcal{U}^{(1)},A^p,\phi^\alpha;P]$.

As above, there is an isomorphism ρ^L

$$\operatorname*{proj}_{\alpha\in A}\left\{C^0(\mathcal{U}^{(1)},A^p,\phi^\alpha;P) \Big/ C^0(\mathcal{U}^{(1)},A^p,\phi^\alpha;P) \cap P\cdot C^0(\mathcal{U}^{(1)},A^q)\right\} \xrightarrow{\ \rho^L\ } \operatorname*{proj}_{\alpha\in A} H(\Omega\cap\vec{\mathcal{W}};\phi^\alpha),$$

where $C^0(\mathcal{U}^{(1)},A^p,\phi^\alpha;P)$ is the space of those $c \in C^0[\mathcal{U}^{(1)},A^p,\phi^\alpha;P]$ with the norms (4.1) bounded by a constant independent of k, i.e. instead of (4.1) we have

$$\|c\|_\alpha = \left[\sum_s \int_{U_s} \|c_s(z)\|^2 \exp - 2\phi^\alpha(z)\, d\lambda(z)\right]^{\frac{1}{2}} < \infty.$$

5. PASSAGE FROM SEMILOCAL TO GLOBAL RESULTS

The main problem of this paper is to extend the semilocally defined functions to a globally defined function, i.e. to show that there is an isomorphism λ:

$$(5.1)\qquad \operatorname*{proj}_{\alpha\in A}\left\{ H[\Omega;\phi^{\alpha}]^{p} \Big/ H[\Omega;\phi^{\alpha}]^{p} \cap P\cdot H[\Omega;\phi^{\alpha}]^{q} \right\} \xrightarrow{\ \lambda\ }$$

$$\xrightarrow{\ \lambda\ } \operatorname*{proj}_{\alpha\in A}\left\{ C^{0}[\mathcal{U}^{(1)},A^{p},\phi^{\alpha};P] \Big/ C^{0}[\mathcal{U}^{(1)},A^{p},\phi^{\alpha};P] \cap P\cdot C^{0}(\mathcal{U}^{(1)},A^{q}) \right\},$$

when H satisfies (3.7) and Ω (3.2) and (3.3). A similar isomorphism λ exists for arbitrary pseudoconvex open sets Ω and for H given by (3.11). We decompose the map λ into continuous restriction maps λ_{α} and we will show that for each α there is an $\alpha' \geq \alpha$ and a kind of continuous inverse $\mu_{\alpha,\alpha'}$ such that the following diagram is commutative:

$$\begin{array}{ccc}
H[\Omega;\phi^{\alpha'}]^{p} \Big/ H[\Omega;\phi^{\alpha'}]^{p} \cap P\cdot H[\Omega;\phi^{\alpha'}]^{q} & \xrightarrow{\ I_{\alpha,\alpha'}\ } & H[\Omega;\phi^{\alpha}]^{p} \Big/ H[\Omega;\phi^{\alpha}]^{p} \cap P\cdot H[\Omega;\phi^{\alpha}]^{q} \\
\downarrow \lambda_{\alpha'} & \nearrow \mu_{\alpha,\alpha'} & \downarrow \lambda_{\alpha} \\
C^{0}[\mathcal{U}^{(1)},A^{p},\phi^{\alpha'};P] \Big/ C^{0}[\mathcal{U}^{(1)},A^{p},\phi^{\alpha'};P] \cap P\cdot C^{0}(\mathcal{U}^{(1)},A^{q}) & \xrightarrow{\ I'_{\alpha,\alpha'}\ } & C^{0}[\mathcal{U}^{(1)},A^{p},\phi^{\alpha};P] \Big/ C^{0}[\mathcal{U}^{(1)},A^{p},\phi^{\alpha};P] \cap P\cdot C^{0}(\mathcal{U}^{(1)},A^{q})
\end{array}$$

where the maps I and I' are the identity maps. The map $\mu_{\alpha,\alpha'}$ is defined with the aid of the following theorem.

THEOREM 5.1. *Let the pseudoconvex open set* Ω *satisfy* (3.2) *and* (3.3), *let the covering* $\mathcal{U}^{(1)}$ *be given as in section* 6.1 *and let* ϕ^{α} *be a plurisubharmonic function on* Ω. *Then there is a* $K > 0$ *and positive constants* N *and* M *depending on the matrix* P *with polynomial entries, such that for each sequence* $\{K_k\}_{k=1}^{\infty}$ *there is a sequence* $\{M_k\}_{k=1}^{\infty}$ *so that for every* $h \in C^{0}[\mathcal{U}^{(1)},A^{p},\phi^{\alpha};P]$ *with* $\|h\|_{\alpha,k} \leq K_k$ *there is a function* $v \in A(\Omega)^{p}$ *and a* $g \in C^{0}(\mathcal{U}^{(1)},A^{q})$ *with*

$$v\Big|_{U_s^{(1)}} - h_s = P\cdot g_s \qquad , \qquad U_s^{(1)} \in \mathcal{U}^{(1)} ,$$

and with

$$\left[\int_{\Omega_k} \|v(z)\|^2 \exp - 2\phi^{\alpha}_{N,M,K}(z)\ d\lambda(z) \right]^{\frac{1}{2}} \leq M_k, \qquad k = 1,2,\ldots ,$$

where the plurisubharmonic function $\phi^{\alpha}_{N,M,K}$ *is determined by* (3.5) *and* (3.6). *When* $h \in C^0(\mathcal{U}^{(1)}, A^p, \phi^{\alpha}; P)$, *i.e.* $\{K_k\}_{k=1}^{\infty}$ *is bounded,* (3.2) *and* (3.3) *need not be satisfied and* $\{M_k\}_{k=1}^{\infty}$ *is bounded too.*

Choose N, M and K larger than the constants N, M and K of theorem 5.1 and also larger than N, M and K of the next theorem. With these N, M and K let for each $\alpha \in A$ $\alpha' \geq \alpha$ be such that $\phi^{\alpha'}_{N,M,K} \leq \phi^{\alpha} + A^{\alpha}$ for some constant A^{α}, then for $h \in C^0[\mathcal{U}^{(1)}, A^p, \phi^{\alpha'}; P]$ define

(5.2) $\qquad \mu_{\alpha,\alpha'}(h) = v$

by theorem 5.1 with α replaced by α'. It follows that $\mu_{\alpha,\alpha'}$ is continuous, because the space between the braces on the right hand side of (5.1) is bornologic. That (5.2) is a good definition follows from the following theorem.

<u>THEOREM 5.2</u>. *Let the pseudoconvex open set* Ω *satisfy* (3.2) *and* (3.3) *and let* ϕ^{α} *be a plurisubharmonic function on* Ω. *There are positive* N *and* M *(depending on the matrix* P *with polynomial entries) and* K, *such that for each sequence* $\{K_k\}_{k=1}^{\infty}$ *there is a sequence* $\{M_k\}_{k=1}^{\infty}$ *so that every* $f \in H[\Omega;\phi^{\alpha}]^p$ *with* $\|f\|_{\alpha,k} \leq K_k$ *which can locally be written as* $f = P\cdot g_{\omega}$, $g_{\omega} \in A(\omega)^q$, $\omega \subset\subset \Omega$, $\Omega = \cup\omega$, *can be written as* $f = P\cdot v$ *for some* $v \in H[\Omega,\phi^{\beta}]^q$, *where* $\phi^{\beta} = \phi^{\alpha}_{N,M,K}$, *with*

$$\|v\|^{N,M,K}_{\alpha,k} \stackrel{\text{def}}{=\!=} \|v\|_{\beta,k} \leq M_k \qquad , \qquad k = 1,2,\ldots ;$$

here the plurisubharmonic function $\phi^{\alpha}_{N,M,K}$ *is determined by* (3.5) *and* (3.6). *Moreover, when* $h \in H(\Omega;\phi^{\alpha})^p$, $\{K_k\}_{k=1}^{\infty}$ *is bounded,* (3.2) *and* (3.3) *need not be satisfied and* $\{M_k\}_{k=1}^{\infty}$ *is bounded, i.e.* $v \in H(\Omega;\phi^{\beta})^q$.

Furthermore, that $I_{\alpha,\alpha'} = \mu_{\alpha,\alpha'} \circ \lambda_{\alpha'}$, follows from theorem 5.1 and theorem 5.2 and that $I'_{\alpha,\alpha'} = \lambda_{\alpha} \circ \mu_{\alpha,\alpha'}$ follows from theorem 5.1. Hence, the diagram is commutative, so that the maps $\{\lambda_\alpha\}_{\alpha\in A}$ define the map λ and the maps $\mu_{\alpha,\alpha'}$ define the inverse λ^{-1}.

In the next section we will give the covering $\mathcal{U}^{(1)}$ and we will prove theorems 5.1 and 5.2.

In Ehrenpreis' formulation a function on $\mathcal{W}$ satisfying all the bounds is extended to one global function satisfying all the bounds at the same time. Here, there is no problem in the semilocal extension, but the passage from semilocally defined functions satisfying all the bounds to globally defined functions yields different global functions for the different bounds. Ehrenpreis requires more conditions and in fact, his result is too strong, as the weaker fundamental principle, formulated here, satisfies quite as well.

6. PROOFS OF THEOREMS 5.1 and 5.2.

6.1. *Coverings*

We construct open coverings $\mathcal{U}^{(\lambda)} = \{U_i^{(\lambda)}\}_{i\in I_\lambda}$, $\lambda = 0,1,2,\ldots$ of the pseudoconvex open set Ω that satisfy the following properties:

(6.1) (i) every $U_i^{(\lambda)}$ is pseudoconvex and $U_i^{(\lambda)} \subset\subset \Omega$;

(ii) there is a positive integer L such that more than L distinct sets in $\mathcal{U}^{(\lambda)}$ have empty intersection;

(iii) the size of $U_i^{(\lambda)}$ satisfies

$$\operatorname{diam} U_i^{(\lambda)} \le \min[b4^{-\lambda}d_i, B4^{-\lambda}],$$

where d_i is the distance from $U_i^{(\lambda)}$ to $\partial\Omega$, and $U_i^{(\lambda)}$ contains a cube whose side for any $z \in U_i^{(\lambda)}$ satisfies

$$\text{side} \ge \min[a4^{-\lambda}d(z,\Omega^c), A4^{-\lambda}],$$

for some constants $a < b$ and $A < B$;

(iv) for each μ $\mathcal{U}^{(\mu+1)}$ is a refinement of $\mathcal{U}^{(\mu)}$ and, moreover, each $U_i^{(\mu)} \in \mathcal{U}^{(\mu)}$ enlarged $2^{\mu-\lambda}$ times with respect to some center in $U_i^{(\mu)}$ is contained in some $U_{j_i}^{(\lambda)} \in \mathcal{U}^{(\lambda)}$ for every

$\lambda = 0,1,\dots,\mu-1$; denote the map ρ between I_μ and I_λ with $\rho(i) = j_i$ by $\rho_{\lambda,\mu}$;

(v) there are positive integers $L_{\lambda,\mu}$ depending on λ and μ $(\mu > \lambda)$ such that for each $j \in I_\lambda$ there are at most $L_{\lambda,\mu}$ indices $i_k \in I_\mu$ with $\rho_{\lambda,\mu}(i_k) = j$, $k = 1,2,\dots,L_{\lambda,\mu}$.

When $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ satisfies (3.2) it follows from property (iii) that

(vi) every set in $\mathcal{U}^{(\lambda)}$ that intersects Ω_k is contained in some Ω_ℓ, where $\ell = \ell(k) > k$ depends on k.

The essential idea for the construction of $\mathcal{U}^{(0)}$ has already been used by WHITNEY in [15], which can be found in HÖRMANDER [5] too.

Divide $\mathbb{C}^n$ into a collection of closed cubes with side 1 (such that the vertices form a rectangular lattice), select those cubes in Ω whose distances to Ω^c are larger than the length $\sqrt{2n}$ of their diagonal and call this collection $\mathcal{U}_0$. Divide the remaining cubes into a collection of cubes of side $\frac{1}{2}$, select those cubes in Ω whose distances to Ω^c are larger than $\frac{1}{2}\sqrt{2n}$ and call this collection $\mathcal{U}_1$. Generally, when the collections $\mathcal{U}_0,\mathcal{U}_1,\dots,\mathcal{U}_{k-1}$ have been defined let $\mathcal{U}_k$ consist of those closed cubes with side $\frac{1}{2}^k$ that are not contained in the union of the cubes of $\bigcup_{\ell=0}^{k-1}\mathcal{U}_\ell$, but that are contained in Ω and whose distances to Ω^c are larger than $\sqrt{2n}/2^k$. Then $\mathcal{U}_0' = \bigcup_{k=0}^{\infty} \mathcal{U}_k$ covers Ω and a cube in $\mathcal{U}_k$ can intersect cubes of $\mathcal{U}_\ell$ only if $\ell = k-1$, k or k+1. Hence $\mathcal{U}_0'$ satisfies property (ii) (with $L = 2^{2n}$) and property (iii) (with $\lambda = 0$, $A = 1$, $B = \sqrt{2n}$, $a = 1/(4\sqrt{2n})$ and $b = 1$).

Define a map α on $\mathcal{U}_0'$ by mapping $U_i' \in \mathcal{U}_0'$ to the enlargement of the interior of U_i' with a factor 3/2, the center kept fixed. Finally, define

$$\mathcal{U}^{(0)} = \{U_i^{(0)} \mid U_i^{(0)} = \alpha U_i',\ U_i' \in \mathcal{U}_0'\}.$$

It is still true that $U_i^{(0)} \cap U_j^{(0)} \neq \emptyset$ if and only if $\alpha^{-1}U_i^{(0)} \cap \alpha^{-1}U_j^{(0)} \neq \emptyset$. Hence, the open covering $\mathcal{U}^{(0)}$ of Ω satisfies properties (i), (ii) (with $L = 2^{2n}$) and (iii) (with $A = 3/2$, $B = \sqrt{2n}\,3/2$, $a = 1/(3\sqrt{2n})$ and $b = 2$) for $\lambda = 0$.

Now let $\mathcal{U}^{(0)},\dots,\mathcal{U}^{(\lambda-1)}$ be defined with the properties (i), (ii), (iii), (iv) and (v) satisfied and let each $\mathcal{U}^{(\mu)}$ consist of open cubes $U_i^{(\mu)}$, such that the collection $\mathcal{U}'_\mu$ of the closed cubes $\alpha^{-1}U_i^{(\mu)}$ covers Ω, $\mu = 0,1,\ldots,\lambda-1$. Define $\mathcal{U}'_\lambda$ as the collection of all the closed cubes obtained by dividing each cube in $\mathcal{U}'_{\lambda-1}$ into 4^{2n} closed cubes. Then define

$$\mathcal{U}^{(\lambda)} = \{U_i^{(\lambda)} \mid U_i^{(\lambda)} = \alpha U'_i,\ U'_i \in \mathcal{U}'_\lambda\}.$$

It is clear that $\mathcal{U}^{(\lambda)}$ satisfies properties (i), (ii) and (iii) and it satisfies (iv), since 2 times a cube $U_i^{(\lambda)} \in \mathcal{U}^{(\lambda)}$ is contained in the cube $U_j^{(\lambda-1)} \in \mathcal{U}^{(\lambda-1)}$, when $\alpha^{-1}U_i^{(\lambda)}$ is one of the 4^{2n} cubes $\alpha^{-1}U_j^{(\lambda-1)}$ had been divided in. Hence (v) is satisfied with $L_{\lambda,\lambda-1} = 4^{2n}$, so that $L_{\lambda,\mu} = (4^{2n})^{\mu-\lambda}$.

6.2. *Some lemmas*

The following lemma is an extension of theorem 4.4.2 in HÖRMANDER [6].

LEMMA 6.1. *Let Ω be an open pseudoconvex set, let $\{\Omega_k\}_{k=1}^\infty$ be an increasing sequence of subsets of Ω satisfying* (3.3) *and let ϕ be a plurisubharmonic function on Ω. For any sequence $\{K_k\}_{k=1}^\infty$ there is a sequence $\{M_k\}_{k=1}^\infty$ such that for every* (0,q+1)*-form* g *with locally square integrable coefficients and with $\bar\partial g = 0$ there is a* (0,q)*-form* u *in Ω with locally square integrable coefficients, so that $\bar\partial u = g$ and for every* $k = 1,2,\ldots$

$$\int_{\Omega_k} \|u(z)\|^2 \frac{\exp - 2\phi(z)}{(1+\|z\|^2)^2}\, d\lambda(z) \le M_k^2\,,$$

provided that for each k

$$\int_{\Omega_k} \|g(z)\|^2 \exp - 2\phi(z)\, d\lambda(z) \le K_k^2.$$

In the above formulation theorem 4.4.2 in HÖRMANDER [6] says that $\{M_k\}_{k=1}^\infty$ is bounded when $\{K_k\}_{k=1}^\infty$ is bounded, while (3.3) need not be satisfied (in fact, if $K_k = K$, then $M_k = K$ for $k = 1,2,\ldots$).

PROOF. Let χ be a convex majorant of the nonnegative function $\tilde{\chi}$

$$\tilde{\chi}(t) = \begin{cases} 0 & \text{for } t < 1 \\ \max[0, \log(2^{k+1} K_{k+1}^2)] & \text{for } k \le t < k+1, \quad k = 1,2,\dots . \end{cases}$$

Then $\psi(z) \overset{\text{def}}{=\!=} \chi(\sigma(z)) \ge 0$ is plurisubharmonic in Ω, so that we may apply theorem 4.4.2 in HÖRMANDER [6] in the domain Ω with the plurisubharmonic function $2\phi+2\psi$. This yields a $(0,q)$-form u in Ω with $\bar\partial u = g$ and with for each k

$$\int_{\Omega_k} \|u(z)\|^2 \frac{\exp - 2\phi(z)}{(1+\|z\|^2)^2} d\lambda(z) \le$$

$$\le e^{2\chi(k)} \int_{\Omega_k} \|u(z)\|^2 \frac{\exp(-2\phi(z)-2\psi(z))}{(1+\|z\|^2)^2} d\lambda(z) \le$$

$$\le e^{2\chi(k)} \int_{\Omega} \|u(z)\|^2 \frac{\exp(-2\phi(z)-2\psi(z))}{(1+\|z\|^2)^2} d\lambda(z) \le$$

$$\le e^{2\chi(k)} \int_{\Omega} \|g(z)\|^2 \exp(-2\phi(z)-2\psi(z))\, d\lambda(z) \le$$

$$\le e^{2\chi(k)} \left\{ \int_{\Omega_m} + \sum_{\ell=m}^{\infty} \int_{\Omega_{\ell+1}\setminus\Omega_\ell} \right\} \|g(z)\|^2 \exp(-2\phi(z)-2\psi(z))\, d\lambda(z) \le$$

$$\le e^{2\chi(k)} \left\{K_m^2 + \sum_{\ell=m}^{\infty} 1/2^{\ell+1}\right\} = e^{2\chi(k)} \left\{K_m^2 + 1/2^m\right\}$$

for arbitrary $m \in \{1,2,\dots\}$. So we may take $M_k = e^{\chi(k)}\sqrt{K_1^2+1/2}$. □

It also follows that, if $\{g_n\}_{n=1}^{\infty}$ is a sequence converging in every norm $\|.\|_k$ to zero, $\{u_n\}_{n=1}^{\infty}$ converges in every norm to zero. This follows from the continuity of a bounded map from a bornological space (here a Fréchet space) into another locally convex space, too.

The following lemma is an extension of proposition 7.6.1 in HÖRMANDER [6]. The proof follows the same lines, only here one has to look more carefully to the estimates near the boundary of Ω.

LEMMA 6.2. *For each sequence* $\{K_k\}_{k=1}^{\infty}$ *there is a sequence* $\{M_k\}_{k=1}^{\infty}$ *such that every cocykel* $c \in C^p[\mathcal{U}^{(\lambda)},A,\phi^\alpha]$, $p \geq 1$, *with* $\|c\|_{\alpha,k} \leq K_k$ *can be written as* $c = \delta c'$ *for some* $c' \in C^{p-1}[\mathcal{U}^{(\lambda)},A,\phi^\beta]$ *with* $\|c'\|_{\beta,k} \overset{\text{not}}{=} \|c'\|_{\alpha,k}^{N,M,0} \leq M_k$ *for every* k, *when*

$$\phi^\beta(z) = \phi^\alpha_{N,M,0}(z) = \phi^\alpha(z) + N\log(1+\|z\|^2) + \log(1+d(z,\Omega^c)^{-M}),$$

where $N = M = \min[p,n]$ *and when the pseudoconvex open set* $\Omega = \bigcup_{k=1}^{\infty}\Omega_k$ *satisfies* (3.2) *and* (3.3). *Moreover, when* $\{K_k\}_{k=1}^{\infty}$ *is bounded,* (3.2) *and* (3.3) *need not be satisfied and* $\{M_k\}_{k=1}^{\infty}$ *is bounded.*

PROOF. Let L_q be the sheaf of germs of (0,q)-forms with locally square integrable coefficients and let Z_q be the subsheaf of $\bar\partial$-closed forms of type (0,q). A section $c \in \Gamma(\Omega,Z_0)$ is a holomorphic function $c \in A(\Omega)$. For $c \in C^p[\mathcal{U}^{(\lambda)},Z_q,\phi^\alpha]$ with $\delta c = 0$ and $\|c\|_{\alpha,k} \leq K_k$ we want to find a $c' \in C^{p-1}[\mathcal{U}^{(\lambda)},Z_q,\phi^\beta]$ such that $\delta c' = c$ and $\|c'\|_{\beta,k} \leq M_k$, when $q = 0$. Assume that this has already been proved for smaller values of p and all q, when $p > 1$, $N = M = p$ and when $\{M_k\}_{k=1}^{\infty}$ depends moreover on p.

We construct a partition $\{\phi_i\}_{i\in I_\lambda}$ of unity subordinate to the covering $\mathcal{U}^{(\lambda)}$ of Ω satisfying for some constant C

$$(6.2) \qquad \|\bar\partial\sqrt{\phi_i(z)}\|^2 \leq \frac{C^2}{\min[1,d(z,\Omega^c)^2]},$$

where

$$\|\bar\partial\phi(z)\|^2 = \sum_{j=1}^{n} |\partial/\partial\bar z_j\ \phi(z)|^2 .$$

For example, let χ be a nonnegative C^∞-function on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ equal to 1 in the closed cube with center 0 and sides 1 and with its support contained in the open concentric cube with sides 3/2. Let the length of the side of $U_i^{(\lambda)} \in \mathcal{U}^{(\lambda)}$ be $3/2\beta_i$ and let the center of $U_i^{(\lambda)}$ be z_i, then define

$$\chi_i(z) = \chi\left(\frac{z-z_i}{\beta_i}\right)$$

and let

$$\phi_i(z) = \frac{\chi_i(z)^2}{\sum\limits_{j\in I_\lambda} \chi_j(z)^2} .$$

By property (6.1)(ii) for each z not more than L terms in the denominator differ from zero and since $\mathcal{U}'_\lambda$ covers Ω at least one term equals 1. Hence, (6.2) follows from this and from property (6.1)(iii). Furthermore, ϕ_i has its support contained in $U_i^{(\lambda)}$.

For $s \in I_\lambda^p$ we set

$$g_s = \sum_{i\in I_\lambda} \phi_i \, c_{is}$$

when $c \in C^p[\mathcal{U}^{(\lambda)}, \mathcal{Z}_q, \phi^\alpha]$. Using $\sum_i \phi_i = 1$, by computing we find $\delta g = c$, if $\delta c = 0$. Furthermore, writing $\phi_i = \sqrt{\phi_i}\,\sqrt{\phi_i}$ and again using $\sum_i \phi_i = 1$ for any function ψ we find

$$\int_{U_s^{(\lambda)}\cap\Omega_k} \|g_s(z)\|^2 \exp - 2\psi(z)\, d\lambda(z) \overset{\text{not}}{=\!=} \|g_s\|^2_{\psi,k} \le$$

$$\le \sum_{i\in I_\lambda} \int_{U_s^{(\lambda)}\cap\Omega_k} \phi_i(z)\, \|c_{is}(z)\|^2 \exp - 2\psi(z)\, d\lambda(z) \le$$

$$\le \sum_{i\in I_\lambda} \|c_{is}\|^2_{\psi,k}.$$

By summing up for each k we get

$$\text{(6.3)} \qquad \|g\|_{\psi,k} \le \|c\|_{\psi,k}$$

for ψ such that the right hand side is finite, hence $g \in C^{p-1}[\mathcal{U}^{(\lambda)}, \mathcal{L}_q, \psi]$.

Let $\bar\partial g = f$ be defined by

$$f_s = \bar\partial g_s = \sum_{i\in I_\lambda} \bar\partial\phi_i \wedge c_{is} = 2 \sum_{i\in I_\lambda} \sqrt{\phi_i}(\bar\partial\sqrt{\phi_i} \wedge c_{is}), \qquad s \in I_\lambda^p .$$

This yields

$$\|f_s\|^{0,1,0}_{\alpha,k} \le 2\left\{\sum_{i\in I_\lambda}\left(\|\bar\partial\sqrt{\phi_i}\wedge c_{is}\|^{0,1,0}_{\alpha,k}\right)^2\right\}^{\frac{1}{2}}$$

and by summing up, in virtue of (6.2) for every k we find

$$\|f\|^{0,1,0}_{\alpha,k} \le 2C\|c\|_{\alpha,k} \le 2C\, K_k,$$

so that $f \in C^{p-1}[\mathcal{U}^{(\lambda)}, Z_{q+1}, \phi^\alpha_{0,1,0}]$.

Now $\delta f = \bar\partial\delta g = \bar\partial c = 0$. If $p > 1$, by the inductive hypothesis (note, that $\phi^\alpha_{N,M,0}$ is plurisubharmonic because Ω is pseudoconvex) there is a cochain $f' \in C^{p-2}[\mathcal{U}^{(\lambda)}, Z_{q+1}, \phi^\alpha_{p-1,p,0}]$ with $\delta f' = f$ and with for every k

$$\|f'\|^{p-1,p,0}_{\alpha,k} \le M'_k\ ,$$

where the sequence $\{M'_k\}^\infty_{k=1}$ depends on $\{2C\, K_k\}^\infty_{k=1}$, hence on $\{K_k\}^\infty_{k=1}$. By lemma 6.1 second part (theorem 4.4.2 of HÖRMANDER [6]) and by property (6.1)(i) for every $s \in I^{p-1}_\lambda$ there is a $(g')_s \in \Gamma(U^{(\lambda)}_s, \mathcal{L}_q)$ so that $\bar\partial(g')_s = (f')_s$ in $U^{(\lambda)}_s$ and

$$\|(g')_s\|^{p,p,0}_{\alpha} \le \|(f')_s\|^{p-1,p,0}_{\alpha}\ .$$

By summing up and by property (6.1)(vi) we get

$$\|g'\|^{p,p,0}_{\alpha,k} \le \|f'\|^{p-1,p,0}_{\alpha,\ell(k)} \le M'_{\ell(k)}\ ,$$

so that $g' \in C^{p-2}[\mathcal{U}^{(\lambda)}, Z_q, \phi^\alpha_{p,p,0}]$.

Finally, set $c' = g - \delta g'$, then for every $k = 1,2,\ldots$ (6.3), property (6.1)(ii) and the above estimate yield

$$\|c'\|^{p,p,0}_{\alpha,k} \le \|c\|^{p,p,0}_{\alpha,k} + p\sqrt{L-p+1}\,\|g'\|^{p,p,0}_{,k} \le$$

$$\le K_k + p\sqrt{L-p+1}\, K'_{\ell(k)} \overset{\text{def}}{=\!=} M_k.$$

Furthermore, $\delta c' = \delta g = c$ and $\bar\partial c' = f-\delta\bar\partial g' = f-\delta f' = f-f = 0$, hence $c' \in C^{p-1}[\mathcal{U}^{(\lambda)}, Z_q, \phi^{\alpha}_{p,p,0}]$.

It remains to consider the case $p = 1$. The fact that $\delta f = 0$ then means that f defines uniquely a $(0,q+1)$-form f in Ω with $\bar\partial f = 0$. By lemma 6.1 there is a $\tilde g \in \Gamma(\Omega, L_q)$ with $\bar\partial \tilde g = f$ and a sequence $\{M'_k\}_{k=1}^{\infty}$ depending on $\{2C\, K_k\}_{k=1}^{\infty}$ with

$$\int_{\Omega_k} \|\tilde g(z)\|^2 \frac{\exp - 2\phi(z)}{(1+\|z\|^2)^2(1+d(z,\Omega^c)^{-2})} d\lambda(z) \le M_k'^2, \qquad k = 1,2,\ldots .$$

Setting $(c')_i = g_i - \tilde g|_{U_i^{(\lambda)}}$ we obtain a cochain with the required properties (using property (6.1)(ii) in the estimate for the cochain $\{\tilde g|_{U_i^{(\lambda)}}\}_{i\in I_\lambda}$).

In fact, there are not more than n induction steps, because all $(0,n)$-forms g satisfy $\bar\partial g = 0$. Therefore, the estimates hold already when p is replaced by $\min[p,n]$ and the sequence $\{M_k\}_{k=1}^{\infty}$ may be taken independent of p.

The second part follows from the second part of lemma 6.1 in case $p = 1$. □

The next lemma is Cartan's theorem B with bounds, it is an extension of theorem 7.6.10 in HÖRMANDER [6]. Let F be either the sheaf of relations of P on Ω, thus $F = R_P$ or the image under P of the sheaf A^q, thus $F = PA^q$.

LEMMA 6.3. *There is a* $K > 0$, *for all polynomial matrices* P *there are positive* N *and* M, *for all nonnegative integers* λ *there is a* $\mu > \lambda$ *(depending moreover on* P*) and for each sequence* $\{K_k\}_{k=1}^{\infty}$ *a sequence* $\{M_k\}_{k=1}^{\infty}$ *(depending moreover on* λ *and* P*), such that every cocykel* $f \in C^p[\mathcal{U}^{(\lambda)}, F, \phi^{\alpha}]$, $p \ge 1$, *with* $\|f\|_{\alpha,k} \le K_k$ *can be written as* $\delta f' = \rho^*_{\lambda,\mu} f$ *(i.e.* $(\delta f')_s = f_{s'}$ *with* $s' = \rho_{\lambda,\mu}(s)$ *for* $s \in I_\mu^{p+1}$*) for some* $f' \in C^{p-1}[\mathcal{U}^{(\mu)}, F, \phi^{\beta}]$ *with* $\|f'\|_{\beta,k} \stackrel{\text{not}}{=} \|f'\|^{N,M,K}_{\alpha,k} \le M_k$, *when the pseudoconvex open set* $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ *satisfies* (3.2) *and* (3.3) *and when* $\phi^{\beta} = \phi^{\alpha}_{N,M,K}$ *is the plurisubharmonic function given by* (3.5) *and* (3.6), *where* ϕ^{α} *is a plurisubharmonic function in* Ω. *Moreover, when* $\{K_k\}_{k=1}^{\infty}$ *is bounded,* (3.2) *and* (3.3) *need not be satisfied and* $\{M_k\}_{k=1}^{\infty}$ *is bounded.*

PROOF. In HÖRMANDER [6] proposition 7.6.5 it is shown that for each $p\times q$-matrix P with polynomial entries there are a number $0 < \delta < 1$ and constants C, $\tilde{N}$ and N' such that, when S denotes the unit cube (actually in HÖRMANDER [6] the unit ball is used, but this only changes the constants), for every $0 < \varepsilon \le 3/2$ and for every $u \in A(S+z/\varepsilon)^q$ there is a $v \in A(\delta S+z/\varepsilon)^q$ with

$$P(\varepsilon w)v(w) = P(\varepsilon w)u(w), \qquad w \in \delta S + z/\varepsilon,$$

and with

$$\sup_{w\in\delta S+z/\varepsilon} \|v(w)\| \le C\varepsilon^{-N'}(1+\|z/\varepsilon\|)^{\tilde{N}} \sup_{w\in S+z/\varepsilon} \|P(\varepsilon w)u(w)\|.$$

Actually this is formula (7.6.5) of HÖRMANDER [6] and it follows from the proof given there, that the constants δ, C, N' and $\tilde{N}$ can be taken independent of ε, if we write $C\varepsilon^{-N'}$ in the above estimate. Therefore, by shrinking the variable w with a factor ε, we find again constants C, $t > 1$, $\tilde{M}$ and $\tilde{N}$ such that for $0 < \eta < 3/2t^{-1}$ and for every $u \in A(t\eta S+z)^q$ there is a $v \in A(\eta S+z)^q$ with

$$P(w)v(w) = P(w)u(w), \qquad w \in \eta S + z$$

and with

$$\sup_{w\in\eta S+z} \|v(w)\| \le C\eta^{-\tilde{M}}(1+\|z\|)^{\tilde{N}} \sup_{w\in t\eta S+z} \|P(w)u(w)\|.$$

Now we change this estimate into one with L^2-norms. Let $V_i \in \mathcal{U}^{(\mu)}$, choose $\nu > \mu$ so that $2^{\nu-\mu} \ge t+1$ and let $U_j \in \mathcal{U}^{(\nu)}$ be such that $\rho_{\mu,\nu}(j) = i$. We write U_j with center z_j and sides η_j as $U_j = \eta_j S+z_j$. Since $\alpha^{-1}U_j \subset \alpha^{-1}V_i$ we have $tU_j = t\eta_j S + z_j \subset \{z \mid \|z-z'\| \le \frac{1}{4}\mathrm{diam}\,\alpha^{-1}V_i+\mathrm{diam}\,U_j\}$ for any $z' \in U_j$ and by property (6.1)(iii) $tU_j \subset \{z \mid \|z-z'\| \le$ $\le (\frac{1}{4}^{\mu+1}+\frac{1}{4}^{\nu})\min[\mathrm{bd}(z',\Omega^c),B]\}$. Therefore, in view of (3.4), $b = 2$, $B = \sqrt{2n}\,3/2$, $\mu \ge 0$ and $\nu \ge 2$ we take $\tilde{K} = \max[\log 8/3,\ 15/32\sqrt{2n}]$ obtaining

$$tU_j \subset \{z \mid z \in S(z';\tilde{K})\}, \qquad z' \in U_j.$$

Also, for $z \in (t+1)U_j$ there is a $z' \in tU_j$ with $\|z-z'\| \le \mathrm{diam}\,U_j$, hence similarly to above

$$(t+1)U_j \subset \bigcup_{z'\in tU_j} S(z';\bar{K})$$

with $\bar{K} = \max[\log 8/7,\ 3/32\sqrt{2n}]$. Now for a plurisubharmonic function ϕ^β, for $N_1 = \tilde{N}/2 + (n+1)/4$, $M_1 = \tilde{M}+n$ and $K_1 = \tilde{K}+\bar{K}$ define the plurisubharmonic function $\phi^{\beta}_{N_1,M_1,K_1}$ by (3.5) and (3.6). In virtue of WLOKA [16] conditions HS_1 and HS_2, property (6.1)(iii) and (3.8) we get

$$\begin{aligned}
&\Big[\int_{U_j} \|v(w)\|^2 \exp - 2\phi^{\beta}_{N_1,M_1,K_1}(w)\, d\lambda(w)\Big]^{\frac{1}{2}} \le \\
&\le C_1\left(\frac{4^{\nu}\eta_j}{a}\right)^{\tilde{M}+n} \sup_{w\in U_j} \|v(w)\| \exp - \phi^{\beta}_{\tilde{N}/2,0,\tilde{K}+\bar{K}}(w) \le \\
&\le C_2(\mu)\eta_j^n \sup_{w\in \eta_j S+z_j} \left(\frac{1+\|z_j\|}{1+\|w\|}\right)^{\tilde{N}} \sup_{w\in t\eta_j S+z_j} \|P(w)u(w)\| \exp - \phi^{\beta}_{0,0,\bar{K}}(w) \le \\
&\le C_3(\mu)\Big[\int_{V_i} \|P(w)u(w)\|^2 \exp - 2\phi^{\beta}(w)\, d\lambda(w)\Big]^{\frac{1}{2}},
\end{aligned} \tag{6.4}$$

where in condition HS_2 (see WLOKA [16]) the radius d_z of the polydisc $D(z,d_z)$ is taken $d_z = \eta_j$ for every $z \in t\eta_j S + z_j$, so that the constant there depends on η_j^{-n} and where

$$\{w \mid w \in D(z,\eta_j),\ z \in t\eta_j S+z_j\} \subset (t+1)\eta_j S+z_j \subset V_i .$$

Conversely to the last lemma this lemma is proved by induction for decreasing p, since the lemma is true for $p \ge L$ (see property (6.1)(ii)), because there are no non-zero cochains $f \in C^L[\mathcal{U}^{(\lambda)},\mathcal{F},\phi^{\alpha}]$. Thus assume that the lemma has been proved for all matrices P, when p is replaced by p+1 and when the constants K, M, N, μ and $\{M_k\}_{k=1}^{\infty}$ depend moreover on p.

In case $\mathcal{F} = \mathcal{R}_P$ there is a polynomial matrix Q, such that $\mathcal{F} = QA^r$ in virtue of lemma 7.6.3 in HÖRMANDER [6] and we can write $f \in C^p[\mathcal{U}^{(\mu)},\mathcal{F},\phi^{\beta}]$ as $f_s = Qg_s$ where $g \in C^p(\mathcal{U}^{(\mu)},A^r)$. Here ϕ^{β} denotes $\phi^{\alpha}_{m',m'',m'''}$ for some m', m'' and m''', In case $\mathcal{F} = PA^q$ we write $Q = P$ and $r = q$, then also $f = Qg$ with $g \in C^p(\mathcal{U}^{(\mu)},A^r)$. Let $\nu \ge \mu + {}^2\log(t+1)$. From (6.4) we obtain a cochain $\breve{g} \in C^p[\mathcal{U}^{(\nu)},A^r,\phi^{\beta}_{N_1,M_1,K_1}]$ with $Q\breve{g}_s = Qg_{s'} = f_{s'}$ where $s' = \rho_{\mu,\nu}(s)$, hence $\rho^*_{\mu,\nu}f = Q\breve{g}$ and with

$$\|\tilde g_s\|_\beta^{N_1,M_1,K_1} \le C_3(\mu)\ \|f_{s'}\|_\beta .$$

Since (3.2) holds property (6.1)(vi) is satisfied and it follows from this property and from property (6.1)(v) that for every k there is an $\ell(k) > k$ with

$$\|\tilde g\|_{\beta,k}^{N_1,M_1,K_1} \le (L_{\mu,\nu})^{p+1}\ C_3(\mu)\ \|f_{s'}\|_{\beta,\ell(k)} \overset{\text{not}}{=\!=} K'_k .$$

When $\delta f = 0$, $\delta Q\tilde g = Q\delta\tilde g = 0$, whence $\delta\tilde g = c$ is a cocykel in $C^{p+1}[\mathcal{U}^{(\nu)},\mathcal{R}_Q,\phi^\beta_{N_1,M_1,K_1}]$.

By the inductive hypothesis we can find $\mu' > \nu$, $N' > N_1$, $M' > M_1$, $K' > K_1$, a sequence $\{M'_k\}_{k=1}^\infty$ belonging to $\{(p+2)\sqrt{L-p-1}\ K'_k\}_{k=1}^\infty$ and a cochain $c' \in C^p[\mathcal{U}^{(\mu')},\mathcal{R}_Q,\phi^\beta_{N',M',K'}]$ with $\delta c' = \rho^*_{\nu,\mu'}c$ and $\|c'\|_{\beta,k}^{N',M',K'} \le M'_k$.

We set $g_0 = \rho^*_{\nu,\mu'}\tilde g - c' \in C^p[\mathcal{U}^{(\mu')},\mathcal{A}^r,\phi^\beta_{N',M',K'}]$ so that $\delta g_0 =$ $= \rho^*_{\nu,\mu'}c - \rho^*_{\nu,\mu'}c = 0$. According to lemma 6.2 there is a sequence $\{M''_k\}_{k=1}^\infty$ belonging to $\{(L_{\nu,\mu'})^{p+1}\ K'_k + M'_k\}_{k=1}^\infty$ and a cochain $g' \in C^{p-1}[\mathcal{U}^{(\mu')},\mathcal{A}^r,\phi^\beta_{N_2,M_2,K'}]$ with $\delta g' = g_0$ and $\|g'\|_{\beta,k}^{N_2,M_2,K'} \le M''_k$ for some $N_2 > N'$ and $M_2 > M'$.

Finally define $f' = Qg' \in C^{p-1}[\mathcal{U}^{(\mu')},\mathcal{F},\phi^\beta_{N_2+N_3,M_2,K'}]$, where N_3 depends on Q. Then $\delta f' = Q\delta g' = Qg_0 = \rho^*_{\nu,\mu'}Q\tilde g = \rho^*_{\nu,\mu'}\rho^*_{\mu,\nu}f = \rho^*_{\mu,\mu'}f$. Furthermore, for every k

$$\|f'\|_{\beta,k}^{N_2+N_3,M_2,K'} \le C'\|g'\|_{\beta,k}^{N_2,M_2,K'} \le C'\ M''_k \overset{\text{def}}{=\!=} M_k .$$

Here $\{M_k\}_{k=1}^\infty$ depends on Q, μ, ν, μ', p and $\{K_k\}_{k=1}^\infty$, but ν depends on μ and P (since t depends on P) and μ' on ν; N_3 depends on Q; N_2 and M_2 depend on p by the inductive hypothesis and on P, since the constants $\tilde N$ and $\tilde M$ depend on P; Q depends on P; C' depends on Q; K' depends on the number of induction steps and finally $\{K'_k\}_{k=1}^\infty$ depends on P and on $\{\|f\|_{\alpha,\ell(k)}^{m',m'',m'''}\}_{k=1}^\infty$. However, there are only finitely many induction steps, so that we can take the largest of all the constants. We start the induction with $p = L$, $\mu = \lambda$, $m' = m'' = m''' = 0$. Therefore, the lemma is true for all p with constants $\{M_k\}_{k=1}^\infty$ depending on P, λ and $\{K_k\}_{k=1}^\infty$; N and M depending on P; μ depending on λ and P; and K fixed.

Moreover, when $\{K_k\}_{k=1}^\infty$ is bounded, i.e. in the above proof we do not use (3.2) and $\{K_k\}$ is bounded, it follows that $\{M'_k\}_{k=1}$ is bounded and by lemma 6.2 (3.2) and (3.3) need not be satisfied and $\{M''_k\}_{k=1}^{\infty}$ is bounded. Hence (3.2) and (3.3) need not be satisfied and $\{M_k\}_{k=1}^{\infty}$ is bounded. □

6.3. *Proof of theorem 5.1.*

Let F be the sheaf PA^q. We can estimate the cocykel $f = \delta h$ in terms of h, then $\|f\|_{\alpha,k} \leq \sqrt{L-1}\, K_k$ and $f \in C^1[U^{(1)},F,\phi^{\alpha}]$. According to lemma 6.3 there is a cochain $f' \in C^0[U^{(\mu)},F,\phi^{\alpha}_{N,M,K}]$ with $\delta f' = \rho^*_{1,\mu}f$ and a sequence $\{M'_k\}_{k=1}^{\infty}$ with $\|f'\|^{N,M,K}_{\alpha,k} \leq M'_k$ for some μ, N, M and K.

For every $i \in I_\mu$ and $z \in U_i^{(\mu)}$ let

$$v_i(z) = h_j(z) - f'_i(z)$$

where $j = \rho_{1,\mu}(i)$. Then $\delta v = \rho^*_{1,\mu}\delta h - \delta f' = \rho^*_{1,\mu}f - \rho^*_{1,\mu}f = 0$, thus $\{v_i \mid i \in I_\mu\}$ determines a function $v \in A(\Omega)^p$. Furthermore, using property (6.1)(v) for every k we obtain

$$\left[\int_{\Omega_k} \|v(z)\|^2 \exp - 2\phi^{\alpha}_{N,M,K}(z)\, d\lambda(z)\right]^{\frac{1}{2}} \leq$$

$$\leq \|v\|^{N,M,K}_{\alpha,k} \leq L_{1,\mu}\|h\|^{N,M,K}_{\alpha,k} + K'_k \leq L_{1,\mu}\, K_k + M'_k \stackrel{\text{def}}{=\!=} M_k.$$

Moreover, if $\{K_k\}_{k=1}^{\infty}$ is bounded, (3.2) and (3.3) need not be satisfied and $\{M'_k\}_{k=1}^{\infty}$ is bounded, so that $\{M_k\}_{k=1}^{\infty}$ is bounded, too.

For $s \in I_1$, let $I'(s) \subset I_\mu$ be the set of those $i \in I_\mu$ with $V_i \stackrel{\text{def}}{=\!=} U_i^{(\mu)} \cap U_s^{(1)} \neq \emptyset$. For each $i \in I'(s)$ and $z \in V_i$ we have

$$v(z)-h_s(z) = h_j(z)-f'_i(z)-h_s(z).$$

This is a holomorphic function in $U_s^{(1)}$ and since $h_j-h_s \in \Gamma(U_j^{(1)} \cap U_s^{(1)},F)$ and also $f'_i \in \Gamma(U_i^{(\mu)},F)$, we obtain

$$v\Big|_{U_s^{(1)}} - h_s \in \Gamma(U_s^{(1)},F).$$

Since the sets V_i and $U_s^{(1)}$ are pseudoconvex (property (6.1)(i)), Cartan's theorem B yields

$$v\Big|_{U_s^{(1)}} - h_s \in P\cdot\Gamma(U_s^{(1)},A^q),$$

that is $v\Big|_{U_s^{(1)}} - h_s = P\cdot g_s$ for some $g \in C^0(\mathcal{U}^{(1)},A^q)$. □

6.4. *Proof of theorem 5.2.*

From Cartan's theorem B it follows that for every $i \in I_0$ $f = Pg_i$ in $U_i^{(0)} \in \mathcal{U}^{(0)}$ with $g \in C^0(\mathcal{U}^{(0)},A^q)$. Let $\nu \geq {}^2\log(t+1)$, then as in lemma 6.3 (6.4) yields a cochain $\tilde{g} \in C^0[\mathcal{U}^{(\nu)},A^q,\phi^\alpha_{N_1,M_1,K_1}]$ with $P\tilde{g}_j = f$ in $U_j^{(\nu)}$ for each $j \in I_\nu$ and with

$$\|\tilde{g}_j\|_\alpha^{N_1,M_1,K_1} \leq C_3(0)\ \|f_{\rho_{0,\nu}(j)}\|_\alpha ,$$

where f is regarded as a cocykel in $C^0[\mathcal{U}^{(0)},A^p,\phi^\alpha]$. Summing over j and using properties (6.1)(ii) and (vi) for each k we get an $\ell(k) > k$ with

$$\|\tilde{g}\|_{\alpha,k}^{N_1,M_1,K_1} \leq C_3(0)\ L_{0,\nu}\ \|f\|_{\alpha,\ell(k)} \leq C_3(0)\ L_{0,\nu}\ K_{\ell(k)} \stackrel{\text{not}}{=\!=} K_k' .$$

Consider the differences c of the functions $\tilde{g}_j$ in the overlaps of the sets $U_j^{(\nu)}$ for $j \in I_\nu$, i.e. $c = \delta\tilde{g}$. Then

$$\|c\|_{\alpha,k}^{N_1,M_1,K_1} \leq 2\sqrt{L-1}\ K_k'$$

and $Pc = P\delta\tilde{g} = \delta f = 0$ and also $\delta c = 0$, hence c is a cocykel in $C^1[\mathcal{U}^{(\nu)},\mathcal{R}_P,\phi^\alpha_{N_1,M_1,K_1}]$.

According to lemma (6.3) and (3.8) there are $\mu > \nu$, $N > N_1$, $M > M_1$, $K > K_1$, a sequence $\{M_k'\}_{k=1}^\infty$ (depending on $\{2\sqrt{L-1}\ K_k'\}_{k=1}^\infty$) and a cochain $c' \in C^0[\mathcal{U}^{(\mu)},\mathcal{R}_P,\phi^\alpha_{N,M,K}]$ with $\delta c' = \rho^*_{\nu,\mu}c$ and with

$$\|c'\|_{\alpha,k}^{N,M,K} \leq M_k' .$$

Finally, for every $s \in I_\mu$ we set $v_s(z) = \tilde{g}_{s'}(z)-c_s'(z)$ for $z \in U_s^{(\mu)}$, where $s' = \rho_{\nu,\mu}(s)$, which defines a function $v \in A(\Omega)^q$, because $\delta v =$

$= \rho^*_{\nu,\mu}\delta\tilde{g} - \rho^*_{\nu,\mu}c = 0$, that satisfies for every k

$$\left[\int_{\Omega_k} \|v(z)\|^2 \exp - 2\phi^{\alpha}_{N,M,K}(z)\, d\lambda(z)\right]^{\frac{1}{2}} \leq$$

$$\leq \|v\|^{N,M,K}_{\alpha,k} \leq L_{\nu,\mu}\ \|\tilde{g}\|^{N,M,K}_{\alpha,k} + M'_k \leq$$

$$\leq L_{\nu,\mu}\ K'_k \stackrel{\text{def}}{=\!=} M_k.$$

If $\{K_k\}^{\infty}_{k=1}$ is bounded, (3.2) need not be satisfied and $\{K'_k\}^{\infty}_{k=1}$ is bounded, hence also (3.3) need not be satisfied and $\{M'_k\}^{\infty}_{k=1}$ is bounded, so that $\{M_k\}^{\infty}_{k=1}$ is bounded.

Furthermore, for every $s \in I_\mu$ in $U^{(\mu)}_s$ we have

$$Pv = Pv_s = P\tilde{g}_{s'} - Pc'_s = f. \qquad \square$$

7. EXTENSION OF THE EHRENPREIS-MARTINEAU THEOREM TO NON-ENTIRE FUNCTIONS

The theorems of this section establish the isomorphism between the spaces of analytic functionals carried by closed convex sets $\bar{\Omega}$ with respect to some class of open neighborhoods of $\bar{\Omega}$ and to some class of weight functions on these neighborhoods and the spaces of functions f of exponential type a(z) holomorphic in open convex cones C in $\mathbb{C}^n$. The class of weight functions on the neighborhoods of $\bar{\Omega}$ determines how f behaves near the vertex of C and conversely. The convex set $\bar{\Omega}$ itself determines the cone C and the type a(z) and conversely. These theorems extend the Ehrenpreis-Martineau theorem (cf. EHRENPREIS [3] and MARTINEAU [11]), where Ω is bounded and $C = \mathbb{C}^n$.

In particular we may take for the convex open cone the open cone $T^C \stackrel{\text{def}}{=\!=} \mathbb{R}^n + i\,C \subset \mathbb{C}^n$, where now C is a convex open cone in $\mathbb{R}^n$. In that case functions f holomorphic in T^C may have boundary values on $\mathbb{R}^n$ like distributions, ultradistributions or hyperfunctions depending on the growth of f near the vertex of C. We recall these various cases (cf. KOMATSU [10]).

Let g be a positive convex C^2-function on $\mathbb{R}^+$ with $g(0) = \infty$ and $g(\infty) = 0$, let $\{C_k\}_{k=1}^{\infty}$ be an increasing sequence of compact subcones of C with union C. Then f, holomorphic in T^C, has a finite order distributional boundary value f(x+i0) if and only if for some m and every k f satisfies

$$(7.1) \qquad |f(z)| \leq K_k \exp m\, g(\|y\|), \qquad \|x\| \leq k, \quad y \in C_k, \quad \|y\| \leq 1,$$

when $g(\rho) = \log\rho^{-1}$.

When $g(\rho)$ is an arbitrary convex function steeper than $\log\rho^{-1}$ for ρ small a holomorphic function f in $\mathbb{R}^n + iC$ has a "finite order" ultradistributional boundary value of Beurling type if and only if f satisfies

$$\forall k, \exists m = m(k),\ \forall \ell, \exists K = K(k,\ell):$$

$$(7.2) \qquad |f(z)| \leq K \exp g\left(\frac{\|y\|}{m}\right), \qquad \|x\| \leq \ell, \quad y \in C_k, \quad \|y\| \leq 1;$$

here "finite order" means that m is independent of ℓ [1]. A function f holomorphic in $\mathbb{R}^n + iC$ has an ultradistributional boundary value of Roumieu type if and only if for every k f satisfies

$$(7.3) \qquad |f(z)| \leq K_k \exp(k\|y\|), \qquad \|x\| \leq k, \quad y \in C_k, \quad \|y\| \leq 1.$$

---

1) An ultradistribution f on $\mathbb{R}^n$ of Beurling type is an element of the dual of a space of C^∞-functions ϕ with compact support satisfying for every $h > 0$ and every $p = 0,1,2,\ldots$

$$(*) \qquad \sup_{x \in \mathbb{R}^n} |D^\alpha \phi(x)| \leq K(h)\, h^p M_p, \quad |\alpha| = p,$$

for a certain sequence $\{M_p\}_{p=1}^{\infty}$ of positive numbers.
Now f satisfies: for every compact set K in $\mathbb{R}^n$ there is a $h = h(K) > 0$ such that for all ϕ satisfying (*) with support in K

$$(**) \qquad |\langle f,\phi\rangle| \leq C(K) \sup_{\alpha,x} \frac{|D^\alpha \phi(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$

Saying that f is of "finite order" means that h in (**) can be taken independent of K. Here $g(\rho) = \log \sup_p \rho^p p!\, M_0/M_p$, cf. KOMATSU [10].

Finally f determines a hyperfunction if it satisfies no condition at all, i.e. if an only if for every k f satisfies

$$(7.4) \qquad |f(z)| \le K_k, \qquad \|x\| \le k, \quad y \in C_k, \quad 1/k \le \|y\| \le 1.$$

and f determines a Fourier hyperfunction (see KAWAI [7]) if for every k f satisfies

$$(7.5) \qquad |f(z)| \le K_k \exp 1/k \|x\|, \quad x \in \mathbb{R}^n, \quad y \in C_k, \quad 1/k \le y \le 1.$$

When $g(\rho) = \rho^{-1/(s-1)}$, $s > 1$, the ultradistributional boundary values are in the Gevrey class of order s and conversely.

We will derive three types of theorems concerning functions f satisfying (7.1), (7.2), (7.3) or (7.4), firstly not only for $\|x\| \le k$, but for all $x \in \mathbb{R}^n$, secondly for $\|x\| \le k$ and thirdly concerning functions f holomorphic in arbitrary convex cones C in $\mathbb{C}^n$ satisfying the same kind of growth conditions near the vertex of C as (7.1), (7.2), (7.3) or (7.4). In the last case one cannot speak of boundary values on the distinguished boundary. The various cases (7.1), (7.2), (7.3) and (7.4) will be treated separately.

Now we determine the behaviour of f for $\|z\|$ large. Regard $\mathbb{C}^n$ as a 2n-dimensional real space $\mathbb{R}^{2n}$ by $z = x+iy \leftrightarrow (y,x)$. Let $a(z) = a(y,x)$ be a convex function in $T^C = \mathbb{R}^n + i\,C$ homogeneous of degree one, i.e. a(z) is determined by its values on pr $T^C = \{z \in T^C \mid \|z\| = 1\}$:

$$a(z) = \|z\| \; a(z/\|z\|).$$

Then moreover, f has to satisfy

$$(7.6) \qquad |f(z)| \le K_k \exp a(z), \qquad \|z\| \text{ large}, \quad z \in T^{C_k}.$$

A convex open cone C in $\mathbb{C}^n$ and a convex homogeneous function a(z) in C determine a closed convex set $\bar{\Omega}$ in $(\mathbb{R}^{2n})'$, which is identified with $(\mathbb{C}^n)'$ by $(\xi,\eta) \leftrightarrow \zeta = \xi+i\eta$, by

$$(7.7) \qquad \bar{\Omega} \overset{\text{not}}{=\!=} \Omega(a,C) = \{\zeta \mid -<(\xi,\eta),(y,x)> \le a(z),\ z \in C\} =$$

$$= \{\zeta \mid -\operatorname{Im} z\cdot\zeta \le a(z),\ z \in C\}.$$

If $a(z)$ can be continuously continued to pr $\bar{C}$, then $\bar{C}$ and C determine the same convex set $\bar{\Omega}$.

Conversely, each closed convex set $\bar{\Omega}$ in $(\mathbb{R}^{2n})'$ determines an open (possibly in some linear subspace of $\mathbb{R}^{2n}$) convex cone C in $\mathbb{R}^{2n}$ and a convex function $a(z)$ in C homogeneous of degree one by: let for $z \in \mathbb{R}^{2n}$ and for some real number a $H(z,a)$ be the affine half space in $(\mathbb{R}^{2n})'$

$$H(z,a) = \{\zeta \mid -<\zeta,z> \le a\};$$

then C is the interior (possibly in some linear subspace of $\mathbb{R}^{2n}$) of the set of all $z \in \mathbb{R}^{2n}$ such that $\bar{\Omega} \subset H(z,a)$ for some real number a depending on z and

$$(7.8) \qquad a(z) = \sup_{\zeta\in\bar{\Omega}} -<\zeta,z>.$$

Note that when $z = x+iy$ and $\zeta = \xi+i\eta$

$$<\zeta,z> = \operatorname{Im} z\cdot\zeta.$$

Saying that C is open in $\mathbb{C}^n$ (hence, that C is not contained in a proper linear subspace) is equivalent with saying that $\bar{\Omega}$ does not contain a straight line. Furthermore, when C is an open convex cone in $\mathbb{R}^n$ and when $a(z)$, defined in T^C, can be continued to a continuous function on $\mathbb{R}^n + i\,\bar{C}_k$, $C_k \subset\subset C$, i.e. $a(0,x)$ exists, then $\Omega(a,T^C)$ is bounded in the imaginary direction, namely

$$\Omega(a,T^C) \subset \{\zeta \mid \|\eta\| \le \max_{\|x\|=1} a(0,x)\}.$$

The weight functions on the neighbourhoods of $\bar{\Omega}$ are determined by the function g as follows: for $\sigma > 0$ let

$$(7.9) \qquad g^*(\sigma) = \min_{\rho>0} (g(\rho)+\sigma\rho),$$

then g^* is positive, $g^*(0) = 0$, $g^*(\infty) = \infty$ and $-g^*$ is a convex function. For each τ there is a positive constant C_τ such that

$$(7.10) \qquad g^*(\sigma) \le \tau\sigma + C_\tau, \quad \sigma > 0,$$

since

$$(7.11) \qquad g(\tau) = \max_{\sigma>0} (g^*(\sigma)-\tau\sigma).$$

Indeed, consider the function

$$\tilde{g}(\tau) = \max_{\sigma>0} \{\min_{\rho>0}[g(\rho)+\sigma(\rho-\tau)]\}.$$

The minimum is attained for ρ satisfying $g'(\rho)+\sigma = 0$ and since $-g'$ is a positive decreasing function, this determines ρ as a function of σ with $d\rho/d\sigma = -1/g''(\rho)$. Then the maximum is attained for σ satisfying $-g'(\rho)/g''(\rho)+\rho-\tau-\sigma/g''(\rho) = 0$ and together with $\sigma = -g'(\rho)$ this yields $\rho = \tau$ and $\sigma = -g'(\tau)$. Substituting these values in the square brackets of $\tilde{g}(\tau)$ one obtains $\tilde{g}(\tau) = g(\tau)$ for every $\tau > 0$.

Depending on whether f satisfies (7.3) or (7.2) the function g has to satisfy:

$$(7.12) \qquad \forall t > 0, \quad \forall M > 0, \quad \exists t' \ge t, \exists A = A(M,t)$$

or

$$(7.13) \qquad \forall t' > 0, \quad \forall M > 0, \quad \exists t, 0 < t \le t', \quad \exists A = A(M,t')$$

so that for every $0 < \rho \le 1$

$$(7.14) \qquad g(t'\rho) + M\log\rho^{-1} \le g(t\rho) + A.$$

In view of (7.9) and (7.11) these conditions are equivalent to similar conditions for the function g^*, namely when (7.12) or (7.13) is satisfied such that for every $\sigma \ge 1$

$$(7.15) \qquad g^*\left(\frac{\sigma}{t'}\right) + M\log\sigma \le g^*\left(\frac{\sigma}{t}\right) + A.$$

In case f satisfies (7.1) $g(\rho) = \log(1+\rho^{-1})$ and $g^*(\sigma)$ satisfies $\log(1+\sigma) \le g^*(\sigma) \le \log(1+\sigma) + A$, where $A = 1+\log 2$.

Finally, it is possible to change the functions g and g^* harmless in order to get plurisubharmonic functions. For, there are a vector $z_0 \in \operatorname{pr} C$, a vector $\zeta_0 \in \operatorname{pr} C^*$ ($C^* = \{\zeta \mid \langle\zeta,z\rangle \ge 0,\ z \in C\}$) and a constant $\delta, 0 < \delta < 1$, such that for every $z \in C$

$$\delta\|z\| \le \langle\zeta_0,z\rangle \le \|z\|$$

and for every $\zeta \in \Omega(a,C)$

$$\delta\|\zeta\| \le \langle\zeta,z_0\rangle \le \|\zeta\|.$$

Therefore, in the sequel we may take $g(t\ \mathrm{Im}\ z\cdot\zeta_0)$ instead of $g(t\|z\|)$ and $g^*(t\ \mathrm{Im}\ z_0\cdot\zeta)$ instead of $g^*(t\|\zeta\|)$, but now $g(t\ \mathrm{Im}\ z\cdot\zeta_0)$ and $-g^*(t\ \mathrm{Im}\ z_0\cdot\zeta)$ are plurisubharmonic functions in C and $\Omega(a,C)$, respectively. Hence, theorem 3.1 may be applied, because (7.14) or (7.15) yields (3.7) or (3.12).

7.1. *Ultradistributions of Roumieu type*

In this section the Ehrenpreis-Martineau theorem is extended to functions f holomorphic in tubular radial domains satisfying (7.6) and having ultradistributions of Romieu type as boundary values, i.e. f also satisfies (7.3). At the same time the theorem is extended to functions holomorphic in arbitrary convex cones in $\mathbb{C}^n$ satisfying an estimate like (7.3) near the vertex and satisfying (7.6) for $\|z\|$ large.

Let C be an open convex cone in $\mathbb{C}^n$ or when we write $T^C = \mathbb{R}^n + iC$, C denotes an open convex cone in $\mathbb{R}^n$ and let a(z) be a convex function in C or T^C homogeneous of degree one. Furthermore, let B_k be the closed ball in $\mathbb{R}^n$ with radius k. Then define the space

$$\text{(7.16)}\qquad \mathrm{Exp}[a,C;g] = \operatorname*{proj}_{k\to\infty} H_p(C_k;\ g(k\|z\|)+a(z)+1/k\|z\|)$$

and in case we consider functions in the tubular radial domain T^C define $\tilde{C}_{k,\ell} = \{z \mid y \in C_k,\ \|x\| \le \ell\|y\|\}$ and $T_\ell^{C_k} = \{B_\ell + iC_k\} \cup \tilde{C}_{k,\ell}$, and let

$$(7.17)\quad \mathrm{Exp}[a,T^C;g] = \operatorname*{proj}_{k\to\infty} H_p(T_k^{C_k};g(k\|y\|)+a(z)+1/k\|z\|).$$

If in the last case a(0,x) exists, it will be shown that (7.17) then equals

$$(7.18)\quad \mathrm{Exp}[a,T^C;g] = \operatorname*{proj}_{k\to\infty} H_p(\mathbb{R}^n+i\,C_k;g(k\|y\|)+a(z)+1/k\|z\|).$$

When g satisfies (7.14) for (7.12), these spaces are independent of $p \in \{1,2,\ldots,\infty\}$ as are the following spaces

$$(7.19)\quad A(a,C;g^*) = \operatorname*{ind}_{k\to\infty} H_p(\Omega(a+1/k,C_k);-g\ (\|\zeta\|/k)),$$

$$(7.20)\quad A(a,T^C;g^*) = \operatorname*{ind}_{k\to\infty} H_p(\Omega(a+1/k,T^{C_k});-g^*(\|\xi\|/k)+k\|\eta\|)$$

and when a(0,x) exists Ω is bounded in the imaginary direction and then we define

$$(7.21)\quad A(a,T^C;g^*) = \operatorname*{ind}_{k\to\infty} H_p(\Omega(a+1/k,T^{C_k});-g^*(\|\xi\|/k)),$$

where Ω is defined by (7.7) and a+1/k denotes the function $a(z) + 1/k\|z\|$.

It follows from (7.11) that $e^{iz\cdot\zeta}$ belongs to the spaces (7.19), (7.20) and (7.21) when $z \in C$ or T^C. Therefore, we can define the Fourier transform $F(\mu)$ of an element μ in the dual of (7.19), (7.20) or (7.21) by

$$(7.22)\quad F(\mu)(z) = \langle\mu_\zeta, e^{iz\cdot\zeta}\rangle.$$

The spaces (7.19), (7.20) and (7.21) are injective compact sequences, hence their duals are Fréchet spaces and they consist of analytic functionals carried by $\bar{\Omega}$ with respect to the neighborhoods $\Omega(a+1/k,C_k)$ and with respect to the weight functions $\exp g^*(\|\zeta\|/k)$.

THEOREM 7.1. *If* g *satisfies* (7.14) *for* (7.12) *the Fourier transformation* (7.22) *is an isomorphism from the strong duals of* (7.19), (7.20) *and* (7.21) *onto the spaces* (7.16), (7.17) *and* (7.18), *respectively.*

PROOF. It follows from (7.11) that F is a bounded (hence continuous) map from A', the dual of (7.20), into Exp (7.17): for $y \in C_k$ and $\xi \in C^*_{k+1}$ we have $-y\cdot\xi \le -\delta_k \|y\|\,\|\xi\|$ for some $\delta_k > 0$ and $\{(C^*_{k+1})^c \cap \operatorname{Re}\Omega_p\} \subset \{\xi \mid \|\xi\| \le A_k\}$ for $p \ge k+2$ and some $A_k > 0$, where Ω_p denotes $\Omega(a+1/p, T^{C_p})$; hence using (7.8) for each k, each $p \ge k+2$ and $z \in B_k + i\,C_k$ we get

$$|\langle\mu_\zeta, e^{iz\cdot\zeta}\rangle| \le K_p \exp \sup_{\zeta\in\Omega_p} \left[g^*\left(\frac{\|\xi\|}{p}\right) - p\|\eta\| - y\cdot\xi - x\cdot\eta\right] \le$$

$$(7.23) \qquad \le K_p \exp\left\{\sup_{\substack{\|\xi\|\ge 0\\ \|\eta\|\ge 0}}\left[g^*\left(\frac{\|\xi\|}{p}\right) - \delta_k\|y\|\,\|\xi\| - (p-k)\|\eta\|\right] + g^*\left(\frac{A_k}{p}\right) + a(z) + \frac{1}{p}\|z\|\right\} \le$$

$$\le K_{p,k} \exp\left[g(p\delta_k\|y\|) + a(z) + \frac{1}{p}\|z\|\right],$$

where $\{K_p\}_{p=1}^{\infty}$ is determined by the bounded set μ belongs to. Now choose $p \ge k/\delta_k$, then there is a constant K'_k with

$$|\langle\mu_\zeta, e^{iz\cdot\zeta}\rangle| \le K'_k \exp[g(k\|y\|) + a(z) + 1/k\|z\|], \qquad z \in B_k + i\,C_k.$$

For $z \in \tilde{C}_{k,\ell}$ and $\|x\| \ge \ell$, hence $\|y\| \ge 1$, we have

$$|\langle\mu_\zeta, e^{iz\cdot\zeta}\rangle| \le K_p \exp\left\{\sup_{\zeta\in\Omega_p}\left[g^*\left(\frac{\|\xi\|}{p}\right) - \varepsilon y\cdot\xi\right] + \sup_{\zeta\in\Omega_p}\left[-(1-\varepsilon)y\cdot\xi - x\cdot\eta\right]\right\} \le$$

$$(7.24) \qquad \le K_p \exp\left\{\sup_{\|\xi\|\ge 0}\left[g^*\left(\frac{\|\xi\|}{p}\right) - \varepsilon\delta_k\|y\|\,\|\xi\|\right] + g^*\left(\frac{A_k}{p}\right) + \varepsilon A_k\|y\| + a((1-\varepsilon)y,x) + \frac{1}{p}\|z\|\right\},$$

where $0 < \varepsilon < 1$ may still be chosen. Since $a(y,x)$ is uniformly continuous on $\operatorname{pr}\{z \mid z + i\varepsilon y \in \tilde{C}_{k,\ell}\}$, there is an $\varepsilon(k,\ell)$ such that for $\varepsilon \le \varepsilon(k,\ell)$

$$a((1-\varepsilon)y,x) \le a(z) + \frac{1}{3k}\|z\|, \qquad z \in \tilde{C}_{k,\ell}.$$

Now take $\varepsilon = \min[1/(3kA_k), \varepsilon(k,\ell)]$ and $p \ge 3k$, then for $z \in \tilde{C}_{k,\ell}$, $\|y\| \ge 1$

$$|\langle\mu_\zeta, e^{iz\cdot\zeta}\rangle| \le K_{k,\ell} \exp[a(z) + 1/k\,\|z\|].$$

When μ is an element of the dual of (7.21) we proceed as in (7.24).

In virtue of the open mapping theorem we only have to show that F is injective and surjective. Using the fundamental principle we will do this for the map F from the dual of (7.20) into (7.17); the other cases are similar. Therefore, we show that the space A (7.20) consists just of those functions in a space S of C^∞-functions that satisfy the Cauchy-Riemann equations, where S consists of functions in Ω_k satisfying the same growth conditions as the functions of A and where the space H of Fourier transforms of elements in S' is known.

Let H be the space of functions in 2n complex variables $\theta = (\theta^1,\theta^2) \in \mathbb{C}^n\times\mathbb{C}^n$ holomorphic in $T(T^C)$, where $T(B)$ for $B \subset \mathbb{C}^n$ denotes $\mathbb{R}^{2n} + i\{(y,x) \mid x + iy \in B\} \subset \mathbb{C}^{2n}$,

$$H = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} H_p(T(T_K^{C_k}); g(k\|\operatorname{Im}\theta^1\|) + a(\operatorname{Im}\theta) + \frac{1}{k}\|\operatorname{Im}\theta\| + m\log(1+\|\theta\|)).$$

When $f \in H$ and $\operatorname{Im}\theta^1 \in \bar{C}$, $f(\operatorname{Re}\theta + i\operatorname{Im}\theta)$ is a tempered distribution in $S'_{\operatorname{Re}\theta}$ and f is the Fourier transform of $g_{\xi,\eta}\exp(-\operatorname{Im}\theta^1\cdot\xi - \operatorname{Im}\theta^2\cdot\eta)$, where $g_{\xi,\eta}$ is a distribution in $\mathcal{D}'_{\xi,\eta}$ with support in the set (see VLADIMIROV [14])

$$O(a,T^C) \stackrel{\text{def}}{=\!=} \{(\xi,\eta) \mid -\operatorname{Im}\theta^1\cdot\xi - \operatorname{Im}\theta^2\cdot\eta \le a(\operatorname{Im}\theta),\ \operatorname{Im}\theta \in T^C\}.$$

The proofs given in VLADIMIROV [14] 26.th.2 do not depend on $\|\operatorname{Im}\theta\|$ for $\|\operatorname{Im}\theta\|$ small and combining these proofs with (7.9) and (7.11) yields that H is the Fourier transform of the space

$$S' = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty}\left[W^m_{\infty,0}(O(a,T^C); -g^*\left(\frac{\|\xi\|}{k}\right)+k\|\eta\|-m\log(1+\|\zeta\|))\right]';$$

here $W^m_p(O;g)$ denotes the space of measurable functions f in the closed set O for which the weak derivatives $D^\alpha f$ exist for $|\alpha| \le m$ such that the norm

$$\left[\sum_{|\alpha|\le m}\int_O |D^\alpha f(x) \exp g(x)|^p\, dx\right]^{1/p} < \infty$$

and when $p = \infty$ f must be a C^m-function in the closed set O in the sense of WHITNEY [15] (here O is convex) with a finite sup-norm, where $\infty,0$ means that moreover $\exp g(x)\ |D^\alpha f(x)| \to 0$ as $x \to \infty$ in O, $|\alpha| \le m$.

Actually, let

(7.25) $$S(m,k) \overset{\text{def}}{=\!=} \operatorname*{ind}_{\ell\to\infty} W_2^m(O(a+1/k,T^{C_k});-g^*\left(\frac{\|\xi\|}{k}\right)+\ell\|\eta\|-m\log(1+\|\zeta\|))$$

which is a DFS*-space, so its dual is known as a projective limit (see KOMATSU [8]) and let

(7.26) $$H[m,k] \overset{\text{def}}{=\!=} \operatorname*{proj}_{\ell\to\infty} H_\infty(T(T_\ell^{C_k});g(k\|\operatorname{Im}\theta^1\|)+a(\operatorname{Im}\theta) + + 1/k\|\operatorname{Im}\theta\|+m\log(1+\|\theta\|)).$$

Then for every k there is a p > k such that F and F^{-1} are continuous maps between

(7.27) $$F: S(m,p)' \longrightarrow H[m+n+1,k] \text{ and } F^{-1}: H[m,p] \longrightarrow S(m+2n+2,k)'$$

(after deriving this for S-spaces with sup-norms one has to apply the embedding maps from spaces with L^2-norms into spaces with sup-norms and conversely, see WLOKA [16]).

Now let

$$A(m,k) = S(m,k) \cap A(\Omega_k), \qquad \Omega_k = O(a+1/k,T^{C_k}),$$

then A(m,k) is a closed linear subspace of S(m,k) and A(m+1,k) is the kernel of the continuous map

(7.28) $$\bar\partial_m: S(m+1,k) \xrightarrow{\bar\partial} S(m,k)^n.$$

In virtue of KOMATSU [8] theorem 15

(7.29) $$A(m+1,k)' \simeq S(m+1,k)'\Big/A(m+1,k)^0 \simeq S(m+1,k)'\Big/\overline{\operatorname{Im}\bar\partial_m^t},$$

where $\overline{\operatorname{Im}\partial_m^t}$ is the closure in S(m+1,k)' of the range of the transposed map of (7.28), which equals the weak * closure because S(m+1,k) is reflexive (cf. TREVES [13] prop. 3.5.4). On the other hand, by inspection one easily sees that the topology of A(m,k) as a closed linear subspace of S(m,k) is also given as an inductive limit over ℓ

$$A(m,k) = \underset{\ell\to\infty}{\text{ind}}\, A(m,k)_\ell \overset{\text{def}}{=\!=} \underset{\ell\to\infty}{\text{ind}}\, W_2^m\left(\Omega_k; -g^*\left(\frac{\|\xi\|}{k}\right) + \right.$$

$$\left. + \ell\|\eta\| - m\log(1+\|\zeta\|)\right) \cap A(\Omega_k),$$

hence A(m,k) is reflexive, it is a DFS*-space. Then the space A (7.20) can be written as

$$A = \underset{m\to\infty}{\text{proj}}\, \underset{k\to\infty}{\text{ind}}\, A(m,k).$$

Since A' is a Fréchet-Montel space, its topology is determined by weakly * converging sequences, denoted by σ,s , so that

$$A' = \underset{m\to\infty}{\text{ind}}\, (\underset{k\to\infty}{\text{ind}}\, A(m,k))'_{\sigma,s}\ .$$

Each weakly * converging sequence in $(\underset{k\to\infty}{\text{ind}}\, A(m,k))'$ converges weakly * in every A(m,k)', k = 1,2,..., hence in every $A(m,k)'_\ell$, ℓ,k = 1,2,... and since the embedding map from $A(m,k)_\ell$ into $A(m,k+1)_{\ell+1}$ is compact, it converges strongly in $\underset{k\to\infty}{\text{proj}}\, A(m,k)'$. Furthermore, $\underset{k\to\infty}{\text{proj}}\, A(m,k)'$ is a Fréchet space (it is an FS-space), so that its topology is determined by convergent sequences. Finally, it is clear that the topology of $\underset{m\to\infty}{\text{ind}}\, \underset{k\to\infty}{\text{proj}}\, A(m,k)'$ is also finer than the one of A', so that we have obtained

$$(7.30) \qquad A' = \underset{m\to\infty}{\text{ind}}\, \underset{k\to\infty}{\text{proj}}\, A(m,k)'.$$

Let $W = (w_1,\dots,w_n)$, $w_j = i\theta_j^1 - \theta_j^2$, then any function $f \in H[m+1,p]$ in the closure of $W\cdot\vec{H}[m,p]$ vanishes on the set where W vanishes and in virtue of theorem 5.2 and (7.14) f can be written as $f = W\cdot\vec{g}$ with $\vec{g} \in H[m+1+N,k]^n$ for N sufficiently large and p > k sufficiently large. It follows from this and from (7.29), (7.30) and (7.27) that the Fourier transformation F

is an isomorphism between

$$(7.31) \qquad F: A' \longrightarrow \underset{m\to\infty}{\text{ind}}\ \underset{k\to\infty}{\text{proj}} \left\{ H[m,k] \Big/ H[m,k] \cap W\cdot\vec{H}[m,k] \right\}.$$

It is clear that $H = \underset{m\to\infty}{\text{ind}}\ \underset{k\to\infty}{\text{proj}}\ H[m,k]$ induces on the set where W vanishes the topology of the space (7.17). Now the theorem follows from the fundamental principle, theorem 3.1, although this is not needed in its full generality: since the variety $V = \{(\theta^1,\theta^2) \mid i\theta^1 = \theta^2\}$ is very simple, a similar process as in theorem 4.4.3 of HÖRMANDER [6], now with lemma 6.1 applied instead of HÖRMANDER [6] theorem 4.4.2, yields the extension of a function in the space (7.17) to a function in any H[m,k], k=1,2,... with m sufficiently large, hence the surjectivity of F. The injectivity follows from theorem 5.2. □

In case a(0,x) exists, $\Omega(a,T^C)$ is bounded in the imaginary direction, hence the spaces (7.20) and (7.21) are equal, so that also the spaces (7.17) and (7.18) are equal.

7.2. *Hyperfunctions*

In view of (7.10) this section is in fact the limit case of section 7.1. as g becomes larger and larger. Only here we get a stronger result, namely we obtain analytic functionals with respect to ε-neighborhoods.

Let $C(k) = C_k \cap \{z \mid \|z\| \geq 1/k\}$ and let

$$(7.32) \qquad \text{Exp}[a,C] = \underset{k\to\infty}{\text{proj}}\ H_p(C(k);a(z)+1/k\|z\|)$$

and when we consider functions in T^C, let $T_\ell^{C(k)} = T_\ell^{C_k} \cap \{z \mid \|y\| \geq 1/k\}$ and let

$$(7.33) \qquad \text{Exp}[a,T^C] = \underset{k\to\infty}{\text{proj}}\ H_p(T_k^{C(k)};a(z)+1/k\|z\|).$$

If in (7.33) a(0,x) exists, this space reduces to

$$(7.34) \qquad \text{Exp}[a,T^C] = \underset{k\to\infty}{\text{proj}}\ H_p(\mathbb{R}^n+i\,C(k);a(z)+1/k\|z\|).$$

These spaces are independent of p, as are the following spaces:

$$\text{(7.35)} \qquad A(a,C) = \operatorname*{ind}_{k\to\infty} H_p(\Omega(a+1/k,C);-1/k\|\zeta\|),$$

$$\text{(7.36)} \qquad A(a,T^C) = \operatorname*{ind}_{k\to\infty} H_p(\Omega(a+1/k,T^C);-1/k\|\xi\|+k\|\eta\|)$$

and when $a(0,x)$ exists

$$\text{(7.37)} \qquad A(a,T^C) = \operatorname*{ind}_{k\to\infty} H_p(\Omega(a+1/k,T^C);-1/k\|\xi\|).$$

Also here the collection $\{e^{iz\cdot\zeta}\}_{z\in C'}$ of functions in ζ belongs to the spaces (7.35), (7.36) and (7.37), where C' is C or T^C, and the Fourier transformation F can be defined by (7.22).

THEOREM 7.2. *The Fourier transformation* (7.22) *is an isomorphism from the strong duals of* (7.35), (7.36) *and* (7.37) *onto the spaces* (7.32), (7.33) *and* (7.34), *respectively*

PROOF. The proof is similar to that of theorem 7.1. The estimates (7.23) and (7.24) are valid for $z \in B_k+i\,C(k)$ or $z \in \tilde{C}_{k,\ell}$, $\|y\| \geq 1$, when $\Omega_p = \Omega(a+1/p,T^C)$, $g^*(\rho)$ is replaced by ρ and the function g is omitted. Instead of (7.25) and (7.26) we define the spaces

$$S(m,k) = \operatorname*{ind}_{\ell\to\infty} W_2^m(O(a+1/k,T^C);-1/\ell\|\xi\|+\ell\|\eta\|-m\log(1+\|\zeta\|))$$

and

$$H[m,k] = \operatorname*{proj}_{\ell\to\infty} H_\infty(T(T_\ell^{C(\ell)});a(\operatorname{Im}\theta)+1/k\ \operatorname{Im}\|\theta\|+m\log(1+\|\theta\|)).$$

Due to the fact that the weight function $\exp 1/\ell\|\xi\|$ only influences the domain $C(\ell)$ and not any weight function in the space H, while the function $g^*(\|\xi\|/k)$ in (7.25) influences both C_k and $g(k\|\operatorname{Im}\theta^1\|)$ in (7.26), we can restrict ourselves to ε-neighborhoods of the set Ω, i.e. the neighborhoods $\{\Omega(a+1/k,T^C)\}_{k=1}^{\infty}$ and, therefore, we get instead of (7.27)

$$F:\ S(m,k+1)' \longrightarrow H[m+n+1,k] \quad \text{and} \quad F^{-1}:\ H[m,k+1] \longrightarrow S(m+2n+2,k)'.$$

Also here (7.30) and (7.31) hold and the theorem follows from the fundamental principle, because the sets $\{\Omega_k = T(T_k^{C(k)})\}_{k=1}^{\infty}$ are equivalent to a system $\{\Omega'_k\}_{k=1}^{\infty}$ (i.e. for every k there is an ℓ with $\Omega'_k \subset \Omega_\ell$ and $\Omega'_\ell \subset \Omega_k$) that satisfies (3.3). For example the sets Ω'_k may be chosen convex such that σ is a convex function. Here the extension of a function in (7.33) to a collection of semilocally defined functions satisfying good bounds may also be obtained by a similar process as in HÖRMANDER [6], theorem 4.4.3. □

REMARK. The second isomorphism in theorem 7.2 equals the first one, if in (7.32) and (7.35) C stands for the cone T^C. Indeed, the sets $\{T_k^{C(k)}\}_{k=1}^{\infty}$ are equivalent to $\{z \mid y \in C(k),\ \|x\| \le k\|y\|\}_{k=1}^{\infty}$. Hence (7.33) equals (7.32) and as reflexive spaces (7.36) and (7.35) are equal, too.

7.3. *Ultradistributions of Beurling type*

An ultradistribution which is the boundary value of a holomorphic function f in T^C that satisfies (7.2) and (7.6) is the Fourier transform of an analytic functional carried by $\Omega(a,T^C)$ with respect to the neighborhoods $\{\Omega(a+1/k,T^{C_k})\}_{k=1}^{\infty}$. However, we will not give this isomorphism, but we treat spaces of a more simple structure, namely we consider Fourier transforms of analytic functionals such that sufficiently small neighborhoods of their carrier are contained in a given open set int $\Omega(a,T^C)$.

Let $\{C^m\}_{m=1}^{\infty}$ be a decreasing sequence of convex cones with intersection C and with $C \subset\subset C^m$ and let $\{a_m\}_{m=1}^{\infty}$ be an increasing sequence convex functions, homogeneous of degree one, each a_m defined on C^m or T^{C^m}, with $a_m(z)+\varepsilon_m \le a_{m+1}(z)$, $z \in$ pr C^{m+1} or pr $T^{C^{m+1}}$, for some $\varepsilon_m > 0$, converging in any point of C or T^C to the convex homogeneous function a on C or T^C. Furthermore, let

$$\text{(7.38)}\qquad \mathrm{Exp}(a,C;g) = \operatorname*{ind}_{m\to\infty} H_p\left(C^m; g\left(\frac{\|z\|}{m}\right)+a_m(z)\right),$$

and when we consider functions in T^C, let

$$\text{(7.39)}\qquad \mathrm{Exp}(a,T^C;g) = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} H_p\left(T_k^{C^m}; g\left(\frac{\|y\|}{m}\right)+a_m(z)\right).$$

If in (7.39) $a_m(0,x)$ exists for every m, this space reduces to

$$\text{(7.40)}\qquad \mathrm{Exp}(a,T^C;g) = \operatorname*{ind}_{m\to\infty} H_p\left(T^{C^m}; g\left(\frac{\|y\|}{m}\right)+a_m(z)\right).$$

If g satisfies (7.14) for (7.13) these spaces are independent of p as are the following spaces:

$$(7.41) \qquad A[a,C;g^*] = \operatorname*{proj}_{m\to\infty} H_p(\Omega(a_m,C^m);-g^*(m\|\zeta\|)),$$

$$(7.42) \qquad A[a,T^C;g^*] = \operatorname*{proj}_{m\to\infty} \operatorname*{ind}_{k\to\infty} H_p(\Omega(a_m,T^{C^m});-g^*(m\|\xi\|)+k\|\eta\|)$$

and when $a_m(0,x)$ exists

$$(7.43) \qquad A[a,T^C;g^*] = \operatorname*{proj}_{m\to\infty} H_p(\Omega(a_m,T^{C^m});-g^*(m\|\xi\|)).$$

It follows from an estimate like (7.23) or (7.24) that the collection $\{e^{iz\cdot\zeta} \mid z \in C^{m'} \text{ or } T^{C^{m'}}\}$ of functions in ζ belongs to one of the spaces in the projective sequence (7.41), (7.42) or (7.43) for some m and every $m' > m$ and moreover, this collection is dense there (which follows from the injectivity of the map F defined by (7.22)), hence the projective limits (7.41) and (7.43) are strict (see FLORET & WLOKA [4]). Therefore, since (7.41) and (7.43) are FS-spaces, their duals are the injective limits of the duals of the spaces in the projective sequences (7.41) and (7.43), respectively, see FLORET & WLOKA [4], §26,2.5. Also we will show that (7.42) is a strict projective limit, that moreover (7.42) is dense in each space

$$(7.44) \qquad \operatorname*{ind}_{k\to\infty} H_p(\Omega(a_m,T^{C^m});-g^*(m\|\xi\|)+k\|\eta\|),$$

$m = 1,2,\ldots$ and that its dual equals

$$(7.45) \qquad A[a,T^C;g^*]' \stackrel{\text{def}}{=\!=} \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} [H_p(\Omega(a_m,T^{C^m});-g^*(m\|\xi\|)+k\|\eta\|)]'.$$

<u>THEOREM 7.3</u>. *When* g *satisfies* (7.14) *for* (7.13), *the Fourier transformation* (7.22) *is an isomorphism from the strong duals of the strict FS-spaces* (7.41) *and* (7.43) *and from the strong dual* (7.45) *of* (7.42) *onto the spaces* (7.38), (7.40) *and* (7.39), *respectively*.

<u>PROOF</u>. First we show that F is an isomorphism between (7.45) and (7.39); the proof of the other cases is similar and is finished then. The proof follows the same lines as the one of theorem 7.1. Instead of (7.25) and (7.26) we define

$$S(m) = \operatorname{ind}_{k\to\infty} W_2^m(O(a_m, T^{C^m}); -g^*(m\|\xi\|)+k\|\eta\| - m\log(1+\|\zeta\|))$$

and

$$H[m] = \operatorname{proj}_{k\to\infty} H_\infty(T(T_k^{C^m}); g\left(\frac{\|\operatorname{Im}\theta^1\|}{m}\right)+a_m(\operatorname{Im}\theta)+m\log(1+\|\theta\|)).$$

Then for every m there is an m' > m with

(7.46) $\quad F: S(m)' \longrightarrow H[m']$ and $F^{-1}: H[m] \longrightarrow S(m')'$.

Like in the proof of theorem 7.1 it can be shown that F is an isomorphism from (7.45) onto

$$\operatorname{ind}_{m\to\infty}\left\{H[m] \Big/ H[m] \cap W\cdot\vec{H}[m]\right\}.$$

Now the theorem follows from the fundamental principle; similarly to the remark made at the end of the proof of theorem 7.1, here the extension of a function in (7.39) to a function in some H[m] can be obtained by HÖRMANDER [6], theorem 4.4.3 and lemma 6.1, too.

Secondly we show that (7.42) is dense in (7.44). We proceed as in the case of finite order distributions, cf. BERENSTEIN & DOSTAL [1]. Let

$$A_F \stackrel{\text{def}}{=\!=} \operatorname{proj}_{m\to\infty} \operatorname{ind}_{k\to\infty} H_p(\mathbb{C}^n; -g^*(m\|\xi\|)+k\|\eta\|)$$

and

$$A \stackrel{\text{def}}{=\!=} \operatorname{ind}_{k\to\infty} \operatorname{proj}_{m\to\infty} H_p(\mathbb{C}^n; -g^*(m\|\xi\|)+k\|\eta\|).$$

The Fourier transforms of elements in A_F and A are C^∞-functions with support in B_k, hence the inductive limits over k are strict. Furthermore, also by means of Fourier transformation it follows that the bounded sets of A_F and A are the same. Clearly the injection from A onto A_F is continuous, so that therefore A_F' is a dense linear subspace of A_F'. Now A' equals

(7.47) $$A' = \operatorname{proj}_{k\to\infty} \operatorname{ind}_{m\to\infty} [H_p(\mathbb{C}^n; -g^*(m\|\xi\|)+k\|\eta\|)]'.$$

Since the transposed map, from $\operatorname{ind}_{m\to\infty} \operatorname{proj}_{k\to\infty} [H_p(\mathbb{C}^n; -g^*(m\|\xi\|)+k\|\eta\|)]'$ into A_F', of the identity map is continuous and since by inspection of the neighborhoods of zero of this space and of the image of A_F' in the space (7.47) one sees that the topology of $\operatorname{ind}_{m\to\infty} \operatorname{proj}_{k\to\infty}[H_p(\mathbb{C}^n; -g^*(m\|\xi\|)+k\|\eta\|)]'$ is also less fine than the topology of A'_F, it follows that

$$(7.48) \qquad A'_F = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} [H_p(\mathbb{C}^n;-g^*(m\|\xi\|)+k\|\eta\|)]'.$$

Furthermore, the space (7.39) is a space of "finite order" ultra-distributions of Beurling type, which can be injectively embedded into the space $F(A'_F)$ of all ultradistributions of "finite order" (this follows from a hyperfunction representation, see KOMATSU [10], and the "edge of the wedge theorem" for hyperfunctions). Therefore, the space

$$(7.49) \qquad \operatorname*{proj}_{k\to\infty} [H_p(\Omega(a_m,T^{C^m});-g^*(m\|\xi\|)+k\|\eta\|)]',$$

being injectively embedded in (7.45), can be injectively embedded into A'_F. Hence A_F is dense in (7.44), which follows if we take $p \neq \infty$, so that (7.49) is the dual of (7.44). Since clearly A_F is contained in (7.42), the space (7.42) is dense in the space (7.44). Therefore, there is a continuous bijection from (7.45) onto the dual of (7.42) and the dual of (7.42) can be injectively embedded into A'_F.

Finally, we show that the topology of (7.45) is also less fine than the topology of the strong dual of (7.42). Let V be an open convex set in the space (7.45); we may take $V = \bigcup_{m=1}^{\infty} V_m$, where V_m is an open convex set in the space (7.49) and where $V_m \subset V_{m+1}$, see FLORET & WLOKA [4]. If $p \neq \infty$ (7.49) is the dual of the regular space (7.44), hence for every m we may take

$$V_m \overset{\text{def}}{=\!=} \Bigg\{\mu \;\Big|\; |\langle\mu,\phi\rangle| < \Big[\sum_{j=1}^{m} \|\phi\|^p_{j,k_j}\Big]^{1/p}, \quad \phi \in (7.44), \text{ where}$$

$$\|\phi\|_{j,k_j} \overset{\text{def}}{=\!=} \Big[\int_{\Omega(a_j,T^{C^j})} |\phi(\zeta)|^p \exp p\{g^*(j\|\xi\|)-k_j\|\eta\|\}\, d\lambda(\zeta)\Big]^{1/p}\Bigg\}$$

for some positive numbers k_j, $j = 1,2,\ldots$. Now let the functions M'_{2m} on $\Omega_{2m} = \Omega(a_{2m},T^{C^{2m}})$ be

$$M'_{2m}(\zeta) = \min_{2j-1\le i\le 2m+1} [-g^*(2m\|\xi\|)+k_i\|\eta\|], \quad \zeta \in \Omega_{2j}\backslash\Omega_{2j-2}, \quad j = 1,\ldots,m$$

and M''_{2m} on Ω_{2m+1} be

$$M''_{2m}(\zeta) = \min_{2j+1\le i\le 2m+1} [-g^*(2m\|\xi\|)+k_i\|\eta\|], \quad \zeta \in \Omega_{2j+1}\setminus\Omega_{2j-1}, \quad j = 0,\dots,m$$

and let M_{2m} be a continuous function on Ω_{2m} with

$$M'_{2m}(\zeta) \le M_{2m}(\zeta) \le M''_{2m}(\zeta), \qquad \zeta \in \Omega_{2m}.$$

Then V_{2m} contains a neighborhood of the form

$$V'_{2m} = \left\{\mu \mid |\langle\mu,\phi\rangle| < \alpha_m \|\phi\|_m, \quad \phi \in (7.42), \text{ where } \|\phi\|_m \overset{\text{def}}{=\!=} \left[\int_{\Omega_{2m}} |\phi(\zeta)|^p \exp - p\, M_{2m}(\zeta)\, d\lambda(\zeta)\right]^{1/p}\right\}$$

for some constant α_m, because (7.42) is dense in (7.44). The set $V' = \bigcup_{m=1}^{\infty} V'_{2m}$ is a neighborhood of zero in the DFS*-space $\underset{m\to\infty}{\text{ind}}[H_p(\Omega_{2m};M_{2m}(\zeta))]'$. Hence V' contains a neighborhood of the form

$$W' = \{\mu \mid |\langle\mu,\phi\rangle| < 1,\ \|\phi\|_m \le K'_m,\ m = 1,2,\dots,\ \phi \in (7.42)\}$$

and this set contains a set of the form

$$W = \{\mu \mid |\langle\mu,\phi\rangle| < 1,\ \|\phi\|_{m,k_m} \le K_m, m = 1,2,\dots,\ \phi \in (7.42)\},$$

which is a neighborhood of zero in the strong dual of (7.42) contained in V. □

Note that the duals of (7.42) and (7.43) are subsets of A'_F, which carry a finer topology than the one induced by A'_F (7.48).

7.4. *Distributions*

The function g in (7.1) does not satisfy condition (7.14), but it satisfies: $\forall m > 0$, $\forall M > 0$, $\exists m' \ge m$, $\exists A = A(M,m)$ so that for every $0 < \rho \le 1$

$$(7.50) \qquad mg(\rho) + M \log \rho^{-1} \le m'g(\rho) + A.$$

This condition restricts the growth of $g(\rho)$ as $\rho \to 0$, but a still more restrictive condition enables us to get analytic functionals carried by closed convex sets with respect to ε-neighborhoods, namely $\forall m$, $\exists M$, $\exists A = A(M,m)$ such that

(7.51) $$mg(\rho/M) \le mg(\rho) + A.$$

The function $g(\rho) = \log(1+\rho^{-1})$ satisfies (7.50) and (7.51) and this yields the case of finite order distributional boundary values. As in §7.3 we consider Fourier transforms of analytic functionals, such that sufficiently small ε-neighborhoods of their carrier are contained in a given open set int $\Omega(a,C)$.

Let $\{a_m(z)\}_{m=1}^{\infty}$ be an increasing sequence of convex homogeneous functions on C or T^C converging in any point of C or T^C to a convex homogeneous function $a(z)$ on C or T^C, respectively, such that $a_m(z)+\varepsilon_m \le$ $\le a_{m+1}(z)$, $z \in \mathrm{pr}\, C$ or $\mathrm{pr}\, T^C$, for some $\varepsilon_m > 0$. Define

(7.52) $$\mathrm{Exp}[a,C;g] = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} H_p(C_k; mg(\|z\|)+a_m(z))$$

and when we consider functions on T^C, define

(7.53) $$\mathrm{Exp}[a,T^C;g] = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} H_p(T_k^{C_k}; mg(\|y\|)+a_m(z)).$$

If in (7.53) $a_m(0,x)$ exists for every m, this space reduces to

(7.54) $$\mathrm{Exp}[a,T^C;g] = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} H_p(\mathbb{R}^n+i\,C_k; mg(\|y\|)+a_m(z)).$$

These spaces are independent of p as are the following spaces

(7.55) $$A[a,C;g^*] = \operatorname*{proj}_{m\to\infty} H_p(\Omega(a_m,C); -mg^*(\|\zeta\|)),$$

(7.56) $$A[a,T^C;g^*] = \operatorname*{proj}_{m\to\infty} \operatorname*{ind}_{k\to\infty} H_p(\Omega(a_m,T^C); -mg^*(\|\xi\|)+k\|\eta\|)$$

and when $a_m(0,x)$ exists

(7.57) $$A[a,T^C;g^*] = \operatorname*{proj}_{m\to\infty} H_p(\Omega(a_m,T^C);-mg\ (\|\xi\|)).$$

Like in §7.3 the spaces (7.55), (7.57) and (7.56) are strict projective limits hence the first two spaces, being FS-spaces, are dense in each space in the projective sequences (7.55) and (7.57), respectively and, moreover, this property also holds for the spaces (7.56), whose strong dual equals

(7.58) $$A[a,T^C;g^*]' = \operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} [H_p(\Omega(a_m,T^C);-mg^*(\|\xi\|)+k\|\eta\|)]'.$$

THEOREM 7.4. *When* g *satisfies* (7.50) *and* (7.51), *the Fourier transformation* (7.22) *is an isomorphism from the strong duals of the strict FS-spaces* (7.55) *and* (7.57) *and from the strong dual* (7.58) *of* (7.56) *onto the spaces* (7.52), (7.54) *and* (7.53), *respectively.*

PROOF. After deriving estimates like (7.23) and (7.24) we apply (7.51) obtaining that F is continuous

$$F\colon [H_p(\Omega(a_m,T^C);-mg^*(\|\xi\|)+k\|\eta\|)]' \longrightarrow \operatorname*{proj}_{k\to\infty} H_p(T_k^{C_k};mg(\|y\|)+a_m(z))$$

or

$$F\colon [H_p(\Omega(a_m,T^C);-mg^*(\|\xi\|)]' \longrightarrow \operatorname*{proj}_{k\to\infty} H_p(\mathbb{R}^n+i\,C_k;mg(\|y\|)+a_{m+1}(z)).$$

Instead of (7.25) and (7.26) we define

$$S(m) = \operatorname*{ind}_{k\to\infty} W_2^m(O(a_m,T^C);-mg^*(\|\xi\|)+k\|\eta\| - m\log(1+\|\zeta\|))$$

and

$$H[m] = \operatorname*{proj}_{k\to\infty} H_\infty(T(T_k^{C_k});mg(\|\operatorname{Im}\theta^1\|)+a_m(\operatorname{Im}\theta)+m\log(1+\|\theta\|)).$$

When $g(\rho) = \log(1+\rho^{-1})$ elements of H[m] have tempered distributional boundary values in S' on the distinguished boundary. Now (7.46) holds, see VLADIMIROV [14], because if in (7.51) M depends on k, this is expressed only in the constant A. The rest follows as in theorem 7.3, while a similar remark, concerning condition (3.3) and the extension to semilocally defined functions, made at the end of the proof of theorem 7.2 can be made here, too. □

8. EXAMPLES OF LOCALIZABLE AU-SPACES

In this section we give some examples of localizable spaces (in the sense that theorem 3.1 holds). Some spaces W such that $H = \mathcal{F}W'$ consists of entire functions are localizable here, but not in the sense of EHRENPREIS [3] (examples 8.1 and 8.2), while others are localizable in [3] but not here, such as $\mathcal{D}'$. That $\mathcal{D}'$ is not localizable here is due to the fact that $-\log(1+\|\zeta\|^2)$ is not plurisubharmonic in $\mathbb{C}^n$. Below (example 8.5) we will see that there are subsets of $\mathcal{D}'$ (with a finer topology than the one induced by $\mathcal{D}'$) which are localizable in our sense. These are the spaces of distributions in $\mathcal{D}'$ whose inverse Fourier transforms have their carrier contained in some unbounded convex open set. More generally, the main examples of spaces W, such that $H = \mathcal{F}W'$ consists of non-entire functions, are spaces of objects, such as distributions, ultradistributions and Fourier hyperfunctions, whose inverse Fourier transforms are concentrated in given unbounded convex sets.

In the following examples it can be easily verified that the occurring weight functions are plurisubharmonic or can be changed into plurisubharmonic functions without damaging the spaces and that where needed conditions (3.2), (3.3) and (3.7) or conditions (3.13) and (3.12) are satisfied.

<u>EXAMPLE 8.1</u>. C^∞-*Functions in an open convex set*

Let $O \subset \mathbb{R}^n$ be an open convex (not necessarily bounded) set and let $\{K_k\}_{k=1}^\infty$ be an increasing sequence of compact convex subsets of O exhausting O. Then the space $\mathcal{E}(O)$ of C^∞-functions in O

$$\mathcal{E}(O) = \operatorname*{proj}_{k\to\infty} W_p^k(K_k;O)$$

($p=1,2,\ldots$ or ∞) is localizable. An AU-structure can be obtained from the isomorphism

$$\mathcal{E}(O) \cong \mathcal{F}[\operatorname*{ind}_{k\to\infty} H_p(\mathbb{C}^n;I_k(\operatorname{Im} z)+k\log(1+\|z\|^2))]',$$

where

$$I_k(y) = \sup_{\xi\in K_k} -y\cdot\xi.$$

EXAMPLE 8.2. C^∞-*Functions on a compact convex set*

Let $K \subset \mathbb{R}^n$ be a compact and convex set, which is the closure of an open set. Then the space $E(K)$ of C^∞-functions on K

$$E(K) = \operatorname*{proj}_{k\to\infty} W^k_\infty(K;0)$$

is localizable. The isomorphism F is given by

$$E(K) \cong F[\operatorname*{ind}_{k\to\infty} H_p(\mathbb{C}^n; I_K(\operatorname{Im} z) + k\log(1+\|z\|^2))]'.$$

EXAMPLE 8.3. *Rapidly decreasing* C^∞-*functions in a convex set*

Let $O \subset \mathbb{R}^n$ be an open, unbounded and convex set not containing a straight line, let $\bar{O}$ be its closure and let $\{\bar{O}_m\}_{m=1}^\infty$ be a sequence of closed convex sets with $\bar{O}_m \subset O_{m+1} \subset O$ and with union O. Then the spaces of rapidly decreasing C^∞-functions in O or on $\bar{O}$,

$$S(O) = \operatorname*{proj}_{m\to\infty} W^m_\infty(\bar{O}_m; -m\log(1+\|\xi\|))$$

and

$$S(\bar{O}) = \operatorname*{proj}_{m\to\infty} W^m_\infty(\bar{O}; -m\log(1+\|\xi\|)),$$

respectively, are localizable. Let C be the open convex cone in $(\mathbb{R}^n)'$ determined by $\bar{O}$ (such that (7.7) and (7.8) hold:

$$\bar{O} = \{\xi \mid -y\cdot\xi \le I_O(y) \overset{\text{def}}{=\!=} \sup_{\xi\in\bar{O}} -y\cdot\xi,\ \forall y \in C\}\ ),$$

let $\{C_k\}_{k=1}^\infty$ be an exhausting sequence of compact subcones of C and let $I_m = I_{O_m}$. Then the isomorphisms F are given by

$$S(O) \cong F[\operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} H_p(T^{C_k}; \log(1+\|y\|^{-2m}) + m\log(1+\|z\|^2) + I_m(y))]'$$

and

$$S(\bar{O}) \cong F[\operatorname*{ind}_{m\to\infty} \operatorname*{proj}_{k\to\infty} H_p(T^{C_k}; \log(1+\|y\|^{-2m}) + m\log(1+\|z\|^2) + I_O(y))]',$$

see VLADIMIROV [14]. The spaces between the square brackets consist of all tempered distributions that are the Fourier transforms of tempered distributions with support in $\bar{O}_m$ or $\bar{O}$, respectively.

Actually, examples 8.1 and 8.2 are particular cases of example 8.3. Similarly, let now $\{O_m\}_{m=1}^{\infty}$ be an increasing sequence of subsets of O such that C is a compact subcone of the cone C^m belonging to O_m: $C \subset\subset C^m \subset\subset C^{m+1}$, then the space of C^{∞}-functions in O, decreasing at infinity as $\exp - g^*(m\|\xi\|)$, $m = 1,2,\ldots$, where g^* satisfies (7.15) for (7.13),

$$S(O;g^*) = \underset{m\to\infty}{\mathrm{proj}}\; W_{\infty}^{m}(\bar{O}_m;-g^*(m\|\xi\|))$$

is localizable; it is the Fourier transform of the dual of

$$\underset{m\to\infty}{\mathrm{ind}}\; H_p\left(T^{C^m};g\left(\frac{\|y\|}{m}\right) + m\log(1+\|z\|^2) + I_m(y)\right).$$

EXAMPLE 8.4. *Analytic functions in convex sets decreasing at infinity*

It follows from theorems 7.1, 7.2, 7.3 and 7.4 that the spaces A of holomorphic functions in int $\Omega(a,C)$ or in certain neighborhoods of $\Omega(a,C)$ decreasing at infinity as $\exp - g^*(m\|\zeta\|)$ for every positive m or as $\exp - g^*(\|\zeta\|/k)$ for some k, where g^* satisfies (7.15) for (7.13) or (7.12), respectively, are localizable, i.e. the spaces (7.19), (7.20), (7.21), (7.35), (7.36), (7.37), (7.41), (7.42), (7.43), (7.55), (7.56) and (7.57). In particular we mention the space of real analytic functions ϕ in $\bar{\Omega} \subset \mathbb{R}^n$, where $\bar{\Omega}$ is a closed convex set in $\mathbb{R}^n$ not containing a straight line and where each ϕ is holomorphic in a complex ε-neighborhood of $\bar{\Omega}$ and satisfies there

$$|\phi(\zeta)| \le M \exp - \varepsilon\|\zeta\| ,$$

cf. (7.37) and example 8.7 below.

EXAMPLE 8.5. *Distributions whose inverse Fourier transforms have their carrier contained in a convex, open set*

Let $\Omega = \mathrm{int}\,\Omega(a,T^C)$ be an open, convex set in $\mathbb{C}^n$ not containing a straight line, such that the open cone belonging to it is of the form $\mathbb{R}^n + iC$, where C is an open convex cone in $\mathbb{R}^n$. Since (as in §7.3) Z_F is dense in the spaces (7.56) and (7.57), the spaces (7.53) and (7.54) consist of *all* finite order distributions in $\mathcal{D}'_F$, whose inverse Fourier transforms have their carrier (with respect to ε-neighborhoods) contained in Ω (by this we mean that sufficiently small ε-neighborhoods of the carrier are contained in Ω). Moreover, if Ω is bounded in the imaginary direction, it follows that all distributions in $\mathcal{D}'$ that are the Fourier transforms of analytic functionals in Z' with carrier contained in Ω, must necessarily be of finite order and a holomorphic function belonging to (7.53) (where now $a_m(0,x)$ exists) belongs to (7.54). We may conclude that the spaces (7.53) and (7.54) of all finite order distributions whose inverse Fourier transforms have their carrier contained in a convex, open set Ω are localizable. The isomorphisms F are given in theorem 7.4.

EXAMPLE 8.6. *Ultradistributions whose inverse Fourier transforms have their carrier contained in a convex set*

The same remarks made for finite order distributions in the last example can be made for "finite order" ultradistributions of Beurling type, too. Also, when $a_m(0,x)$ and $a(0,x)$ exists, holomorphic functions belonging to (7.39) or (7.17) belong already to (7.40) or (7.18), respectively. We now consider carriers $\Omega(a,T^C)$ with respect to the neighborhoods $\{\Omega(a+1/k,T^{C_k})\}_{k=1}^{\infty}$ and when we say that a carrier is contained in a certain open set, we mean that these neighborhoods for $k \geq k_0$ are contained in this set. In virtue of (7.15) the spaces (7.39) and (7.40) of all "finite order" ultradistributions of Beurling type, whose inverse Fourier transforms have their carrier contained in a given open convex set Ω (Ω as in example 8.5) are localizable. The isomorphisms F are given in theorem 7.3. Similarly, the spaces (7.17) and (7.18) of all ultradistributions of Roumieu type whose inverse Fourier transforms are carried by a given closed convex set $\bar{\Omega}$ (Ω as in example 8.5) are localizable. The isomorphisms F are given in theorem 7.1.

EXAMPLE 8.7. *Fourier hyperfunctions whose inverse Fourier transforms have their support contained in a closed convex set*

When $a(0,x)$ exists a holomorphic function belonging to (7.33) already belongs to (7.34). As in the case of ultradistributions of Roumieu type we may conclude that the spaces (7.33) and (7.34) of hyperfunctions whose inverse Fourier transforms are carried (with respect to ε-neighborhoods) by closed convex sets $\bar{\Omega}$ (Ω as in example 8.5) are localizable. However, we will consider a particular case of (7.34) consisting of Fourier hyperfunctions. Then the Fourier transforms are again Fourier hyperfunctions (similarly to the case of tempered distributions).

Let $\bar{\Omega} = \Omega(a,T^C)$ be a closed convex set in $\mathbb{R}^n$ not containing a straight line, then $a(y)$ is a convex homogeneous function on $C \subset \mathbb{R}^n$. By $\Omega(\varepsilon)$ we denote a complex ε-neighborhood of $\bar{\Omega}$. Fourier hyperfunctions are hyperfunctions in $\mathbb{R}^n$ represented by means of functions f satisfying an estimate like (7.5), see KAWAI [7]. In [7] it is shown that the space of analytic functionals carried by $\bar{\Omega}$ with respect to ε-neighborhoods and the weight functions $\{\exp \varepsilon\|\xi\|\}_{\varepsilon>0}$ is isomorphic to the space of Fourier hyperfunctions with support in $\bar{\Omega}$ [1]. Theorem 7.2 yields the isomorphism $\mathcal{F}$

$$H(a,C) \overset{\text{def}}{=\!=} \operatorname*{proj}_{k\to\infty} H_p(\mathbb{R}^n + i\,C(k); a(\operatorname{Im} z)+1/k\|z\|) \simeq \mathcal{F}[\operatorname*{ind}_{\varepsilon\to 0} H_p(\Omega(\varepsilon); -\varepsilon\|\xi\|)]',$$

---

1) Actually, in KAWAI [7] the neighborhoods of $\Omega(a,T^C)$ are of the form $\{\Omega(1+1/k,T^{C_k})\}_{k=1}^{\infty}$, but as in theorem 7.2 it can be shown that the dual of

$$(*) \quad \operatorname*{ind}_{k\to\infty} H_p(\Omega(a+1/k,T^{C_k}); -1/k\|\xi\|)$$

is isomorphic to the corresponding space (7.34), which is also isomorphic to the dual of

$$(**) \quad \operatorname*{ind}_{k\to\infty} H_p(\Omega(1/k); -1/k\|\xi\|)$$

according to theorem 7.2. It is even more simple to show the first isomorphism (which is done in KAWAI [7], when a is constant) directly with the aid of HÖRMANDER [6] th. 4.4.2 instead of lemma 6.1. Hence, both spaces of analytic functionals are the same with respect to ε-neighborhoods or to the neighborhoods $\{\Omega(a+1/k,T^{C_k}\}_{k=1}^{\infty}$. Being reflexive spaces (namely DFS-spaces) the spaces (*) and (**) are equal, i.e. a holomorphic function ϕ in $\Omega(\varepsilon)$ satisfying

$$|\phi(\zeta)| \le M \exp - \varepsilon\|\xi\|, \qquad \zeta \in \Omega(\varepsilon),$$

is holomorphic in some larger set $\Omega(\varepsilon) \cup \Omega(a+1/k,T^{C_k})$ and satisfies there

$$|\phi(\zeta)| \le M' \exp - \delta\|\xi\|$$

for some $\delta \le \varepsilon$ depending on C_k and ε and for some M' depending on M, ε and C_k.

where in view of (7.5) H(a,C) is a space of Fourier hyperfunctions. Hence, it is the space of all Fourier hyperfunctions whose inverse Fourier transforms have their support contained in $\bar{\Omega} \subset \mathbb{R}^n$ and it is localizable. In particular, H(a,C) is an AU-space of Fourier hyperfunctions.

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